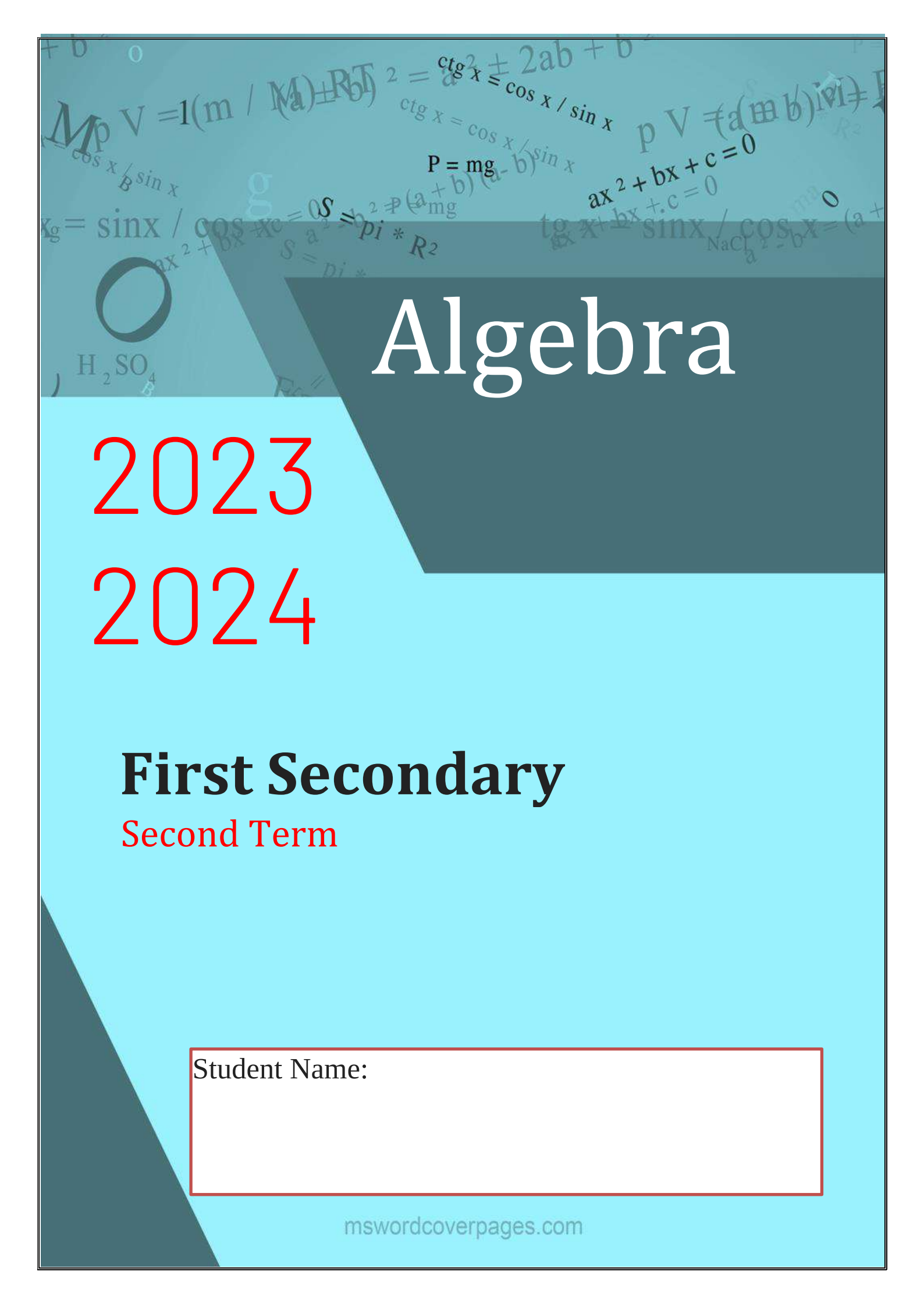




Algebra

2023
2024

First Secondary
Second Term

Student Name:



Unit 1

**Lesson (1-1)****Organizing data in Matrices****Lesson objectives****Related Links**

- ✚ Identify matrix.
- ✚ Define some special matrices.
- ✚ Recognize Symmetric and skew symmetric matrix.
- ✚ Identify Equality of two matrices.
- ✚ Find Multiply a real number by a matrix.



The concept of a “matrix” is one of the longest-running and well-developed ideas in all of mathematics. Having their origins as far back as 2 BC with Chinese mathematicians, matrices are a central object of study in mathematics and are used frequently in other disciplines such as physics and computer science. As well as the applications of matrices to other science disciplines, the properties of matrices are interesting for their own reasons and, as such, are also studied within mathematics in a more theoretical and abstract framework.

There are many ways to motivate the study of matrices, such as through systems of linear equations or through geometric transformations, but these approaches rely on first establishing a correct understanding of what matrices are. A matrix is a rectangular array which is separated into rows (which run horizontally) and columns (which run vertically). Each “entry” in a matrix is then described with respect to the row and column that it appears in, which is defined as follows.

First row	75	135	150	215
Second row	100	168	210	282
Third row	80	100	144	64
	↑	↑	↑	↑
	First column	Second column	Third column	Fourth column

This form is called a "**Matrix**", the numbers enclosed by two parentheses () are called "**elements of the matrix**"

- This matrix has 3 rows and 4 columns, thus it is said that it is a matrix of order 3×4 or simply "**a 3×4 matrix**". You always write the number of rows first and the number of columns second.

We notice that: number of elements of the matrix = $3 \times 4 = 12$ elements.



Remarks

- ① The matrix is denoted by one of capital letters as : A , B , C , X , ...
- ② If the order of the matrix is $m \times n$, then the number of elements of the matrix = no. of rows $\times$ no. of columns = $m \times n$
- ③ If A is a matrix of order $m \times n$, then we can express it by the form:
 $A = (a_{ij})$ where: $i = 1, 2, 3, \dots, m$ $j = 1, 2, 3, \dots, n$

Example ①

A leather factory produces three kinds of leather products: shoes, bags and belts. If the factory has two branches A and B and the average of the daily production from each kind and each branch is as follows:

The branch A: products 110 shoes, 200 bags and 150 belts.

The branch B: products 140 shoes, 180 bags and 120 belts.

Put the previous data in the form of a matrix and mention the order of matrix.

Solution

$$X = \begin{pmatrix} 110 & 200 & 150 \\ 140 & 180 & 120 \end{pmatrix} \text{ of order } 2 \times 3 \quad \text{OR} \quad Y = \begin{pmatrix} 110 & 140 \\ 200 & 180 \\ 150 & 150 \end{pmatrix} \text{ of order } 3 \times 2$$

Example ②

Write by listing

- ① The matrix : $A = (a_{ij})$ where : $i = 1, 2, 3$ and $j = 1, 2$
- ② The matrix : $B = (b_{ij})$ where : $i = 1$ and $j = 1, 2, 3$

Solution

$$\textcircled{1} A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \quad \textcircled{2} B = (b_{11} \quad b_{12} \quad b_{13})$$

Example ③

$$\text{If } A = \begin{pmatrix} -2 & 0 & \frac{1}{2} \\ 1 & 6 & -4 \end{pmatrix}, B = \begin{pmatrix} 4 & 5 \\ -1 & 8 \end{pmatrix} \text{ and } C = \begin{pmatrix} 3 & -3 & \sqrt{5} \\ -1 & 0 & -2 \\ 2 & \frac{1}{4} & 9 \end{pmatrix}$$

First: Write the order of each of A , B and C

Second: Write the following elements: a_{22} , b_{21} , a_{23} , c_{32} , c_{23} , c_{13}

Solution

First: A of order 2×3 B of order 2×2 C of order 3×3

Second:

$a_{22} = 6$	$b_{21} = -1$	$a_{23} = -4$	$c_{32} = \frac{1}{4}$	$c_{23} = -2$	$c_{13} = \sqrt{5}$
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**Example 4**

Write the matrix (a_{ij}) in the order 3×2 where : $a_{ij} = 2i - j$

Solution

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \quad \therefore A = \begin{pmatrix} 2 \times 1 - 1 & 2 \times 1 - 2 \\ 2 \times 2 - 1 & 2 \times 2 - 2 \\ 2 \times 3 - 1 & 2 \times 3 - 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{pmatrix}$$

Some special matrices

- A Square matrix:** It is a matrix in which the number of its rows equals the number of its columns. For example: $\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$ (a 2×2 square matrix)
- B Row matrix:** It is a matrix containing one row and any number of columns. For example : $(2 \ 4 \ 6 \ 8)$ (a 1×4 row matrix)
- C Column matrix:** It is a matrix containing one column and any number of rows. For example: $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ (a 3×1 column matrix)
- D Zero matrix:** It is a matrix in which all of its elements are Zeros. It may be a square matrix or not. For examples:
 (0) is a 1×1 zero matrix, $(0 \ 0)$ is a 1×2 zero matrix, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a 2×1 zero matrix, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a 2×2 square zero matrix and is denoted by $\bigcirc$.
- E Diagonal matrix:** It is a square matrix in which all elements are zeros except the elements of its diagonal then at least one of them is not equal to zero. For example:
the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (is a 3×3 diagonal matrix)
- F Unit matrix:** it is a diagonal matrix in which each element on the main diagonal has the numeral 1, while 0 exists in all other elements , it is denoted by I. for example:
each of:

$$(1), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is a unit matrix.}$$



$$(A^t)^t = A$$

Matrix transpose

If $A = (a_{ij})$, then $A^t = (a_{ji})$

Replacing columns by rows

Example 5

Find the transpose of each of the following matrices, and then mention the order of the matrix transpose.

$$A = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 9 \\ 1 \\ -6 \end{pmatrix}, \quad C = (3 \quad -5 \quad 0)$$

$$D = \begin{pmatrix} 5 & -6 \\ 7 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} -3 & 1 & 0 \\ 5 & -6 & 4 \\ 7 & -2 & 3 \end{pmatrix}$$

Solution

$$A^t = \begin{pmatrix} 2 & -1 \\ 3 & 5 \\ 0 & 6 \end{pmatrix} \quad \text{of order } 3 \times 2 \quad B^t = (9 \quad 1 \quad -6) \quad \text{of order } 1 \times 3$$

$$C^t = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} \quad \text{of order } 3 \times 1 \quad D^t = \begin{pmatrix} 5 & 7 \\ -6 & -1 \end{pmatrix} \quad \text{of order } 2 \times 2$$

$$E^t = \begin{pmatrix} -3 & 5 & 7 \\ 1 & -6 & -2 \\ 0 & 4 & 3 \end{pmatrix} \quad \text{of order } 3 \times 3$$

Example 6

Write the transpose of each of the following matrices, then show that what you deduce.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution

$$A^t = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A \quad \therefore O^t = O$$

$$B^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B \quad \therefore I^t = I$$

**Example 7**

If $A = \begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & -3 \\ 3 & 5 & -2 \end{pmatrix}$ check that $(A^t)^t = A$

Solution

$$\therefore A = \begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & -3 \\ 3 & 5 & -2 \end{pmatrix}$$

$$\therefore A^t = \begin{pmatrix} 4 & 1 & 3 \\ 2 & 0 & 5 \\ 1 & -3 & -2 \end{pmatrix}$$

$$\therefore (A^t)^t = \begin{pmatrix} 4 & 2 & 1 \\ 1 & 0 & -3 \\ 3 & 5 & -2 \end{pmatrix}$$

$$\therefore (A^t)^t = A$$

The equality of two matrices

matrix A = matrix B if : $a_{ij} = b_{ij}$ for each i and for each j

Example 8

① Find the value of each of x, y and z if: $\begin{pmatrix} z & 0 & 2 \\ 4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 0 & x+5 \\ 4 & 2y-3 & 5 \end{pmatrix}$

Solution

$$\therefore x + 5 = 2$$

$$\therefore x = 2 - 5$$

$$\therefore x = -3$$

$$\therefore 2y - 3 = 7$$

$$\therefore 2y = 7 + 3$$

$$\therefore y = \frac{10}{2} = 5$$

$$\therefore z = -1$$

② If: $X = \begin{pmatrix} 3a+1 & 12-b & e^3 \\ c+2d & 18 & 6 \end{pmatrix}$ and $Y = \begin{pmatrix} 7 & 9 \\ 3 & 18 \\ -8 & 2c+d \end{pmatrix}$

Find: a, b, c, d and e if you know that : $X = Y^T$

Solution

$$\therefore X = Y^T$$

$$\therefore \begin{pmatrix} 3a+1 & 12-b & e^3 \\ c+2d & 18 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 3 & -8 \\ 9 & 18 & 2c+d \end{pmatrix}$$

$$\therefore 3a + 1 = 7$$

$$\therefore 12 - b = 3$$

$$\therefore c + 2d = 9$$

$$\therefore 2c + d = 6$$

$$\therefore e^3 = -8$$

$$\therefore 3a = 7 - 1$$

$$\therefore b = 12 - 3$$

By solving the 2 equations:

$$\therefore e = \sqrt[3]{-8}$$

$$\therefore 3a = 6$$

$$\therefore b = 9$$

$$\therefore c = 1$$

$$\& \therefore d = 4$$

$$\therefore e = -2$$

$$\therefore a = 2$$

Symmetric and Skew Symmetric Matrices

If A is a square matrix, then it is called a symmetric if and only if $A = A^t$, and is called skew symmetric if and only if $A = -A^t$

**Example 9**

Is the matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 5 \\ -1 & 5 & 6 \end{pmatrix}$ Symmetric or skew symmetric?

Solution

$$\therefore A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 5 \\ -1 & 5 & 6 \end{pmatrix}$$

$$\therefore A^t = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 5 \\ -1 & 5 & 6 \end{pmatrix} = A$$

$\therefore A$ is a symmetric

Example 10

Is the matrix $B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 5 \\ 1 & -5 & 0 \end{pmatrix}$ Symmetric or skew symmetric?

Solution

$$\therefore B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 5 \\ 1 & -5 & 0 \end{pmatrix}$$

$$\therefore B^t = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -5 \\ -1 & 5 & 0 \end{pmatrix}$$

$$\therefore B^t = -1 \times \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 5 \\ 1 & -5 & 0 \end{pmatrix} = -B$$

$$\therefore B^t = -B \quad \& \quad -B^t = B$$

$\therefore B$ is a skew symmetric



PRACTICE

Choose the correct answer

Q (1): Which of the following is the matrix $A = (a_{ij})$ where $i = 1$ and $j = 1, 2$?

- (a) $\begin{bmatrix} a_{12} & a_{11} \end{bmatrix}$
 (b) $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$
- (c) $\begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$
 (d) $\begin{bmatrix} a_{12} \\ a_{11} \end{bmatrix}$

Q (2): Which of the following is the matrix $A = (a_{ij})$ where $i = 1, 2$ and $j = 1, 2, 3$?

- (a) $\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$
 (b) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$
- (c) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$
 (d) $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$

Q (3): Find the matrix $A = (a_{xy})$ with an order of 3×3 , whose elements are given by the formula $a_{xy} = 5x + 4y$

- (a) $\begin{bmatrix} 9 & 13 & 17 \\ 14 & 18 & 22 \\ 17 & 22 & 27 \end{bmatrix}$
 (b) $\begin{bmatrix} 9 & 13 & 17 \\ 14 & 18 & 22 \\ 19 & 23 & 27 \end{bmatrix}$
- (c) $\begin{bmatrix} 9 & 14 & 19 \\ 13 & 18 & 22 \\ 17 & 22 & 27 \end{bmatrix}$
 (d) $\begin{bmatrix} 9 & 14 & 19 \\ 14 & 18 & 22 \\ 19 & 23 & 27 \end{bmatrix}$

Q (4): Given that: A is a matrix of order 3×2 , where $a_{11} = 0$, $a_{21} = 4$, $a_{22} = \frac{1}{2} a_{11}$, $a_{31} = 8$ and $a_{32} = \frac{1}{4} a_{21}$, determine A

- (a) $\begin{bmatrix} 0 & 5 \\ 4 & 0 \\ 8 & 0 \end{bmatrix}$
 (b) $\begin{bmatrix} 0 & 5 & 0 \\ 4 & 8 & 0 \end{bmatrix}$
- (c) $\begin{bmatrix} 0 & 8 \\ 4 & 0 \\ 8 & 1 \end{bmatrix}$
 (d) $\begin{bmatrix} 5 & 0 \\ 0 & 4 \\ 1 & 8 \end{bmatrix}$



Q (5):

The table below represents the prices of some drinks in a café. The café owner changes the prices of the drinks, so that each drink is now two times its original price. Determine the matrix that represents the new prices of the drinks.

Drink	Small Size	Big Size
Strawberry Juice	1.5	5.5
Orange Juice	2	8.5
Mango Juice	3.5	9

Ⓐ $\begin{bmatrix} 3.5 & 7.5 \\ 4 & 10.5 \\ 5.5 & 11 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 4.5 & 16.5 \\ 6 & 25.5 \\ 10.5 & 27 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 1.5 & 5.5 \\ 2 & 8.5 \\ 3.5 & 9 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 3 & 11 \\ 4 & 17 \\ 7 & 18 \end{bmatrix}$

Q (6):

The coach of a school basketball team recorded the statistics of three players in a tournament as follows: Jacob: 7 matches, 15 shots, and 5 goals; Jack: 13 matches, 24 shots, and 8 goals; and, Thomas: 16 matches, 23 shots, and 19 goals. Arrange these data in a matrix where the rows represent the players from least goals to most, the first column gives the number of matches, the second column gives the number of shots, and the third column gives the number of goals.

Ⓐ $\begin{bmatrix} 7 & 15 & 5 \\ 16 & 23 & 19 \\ 13 & 24 & 8 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 7 & 15 & 5 \\ 13 & 23 & 8 \\ 16 & 24 & 19 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 7 & 15 & 5 \\ 13 & 24 & 8 \\ 16 & 23 & 19 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 16 & 23 & 19 \\ 13 & 24 & 8 \\ 7 & 15 & 5 \end{bmatrix}$

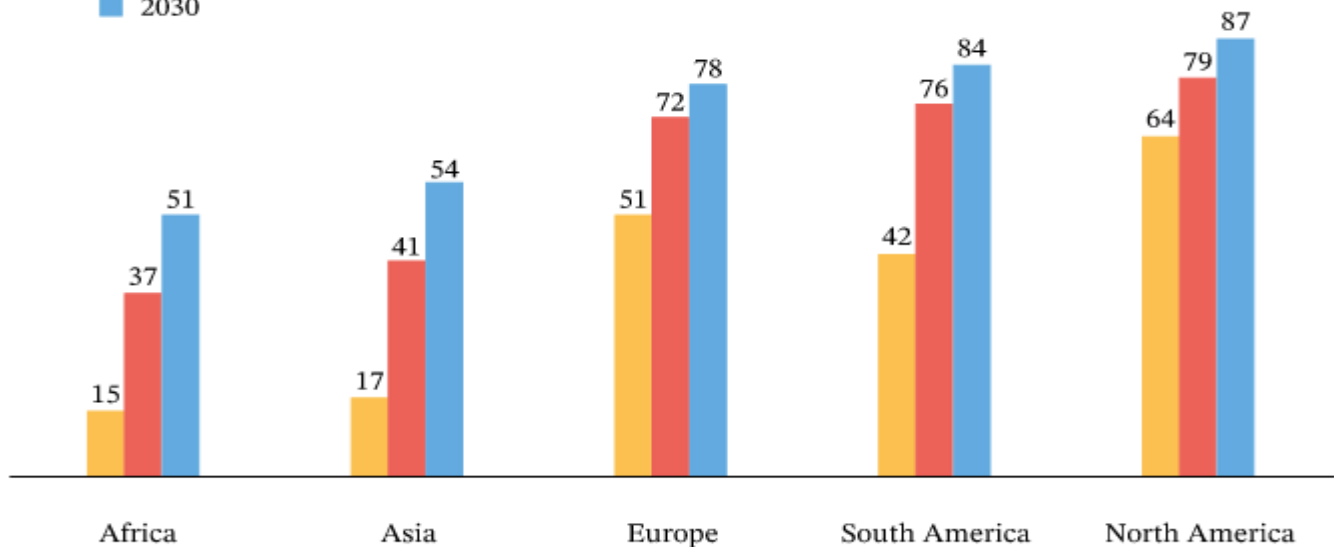


Q (7):

The given bar chart shows actual and predicted data for several continents about the percent of the population who live in urban areas. Represent this data in a matrix where the rows represent years in order from past to future, and the columns represent the continents in alphabetical order.

1950
2007
2030

Percentage of Population in Urban Areas



- (a) $\begin{bmatrix} 0.29 & 0.49 & 0.60 & 0.15 & 0.37 & 0.51 \\ 0.17 & 0.41 & 0.54 & 0.51 & 0.72 & 0.78 \\ 0.42 & 0.76 & 0.84 & 0.64 & 0.79 & 0.87 \end{bmatrix}$
- (b) $\begin{bmatrix} 0.15 & 0.17 & 0.51 & 0.42 & 0.64 \\ 0.37 & 0.41 & 0.72 & 0.76 & 0.79 \\ 0.51 & 0.54 & 0.78 & 0.84 & 0.87 \end{bmatrix}$
- (c) $\begin{bmatrix} 1.5 & 5.1 & 1.7 & 4.2 & 6.4 \\ 3.7 & 7.2 & 4.1 & 7.6 & 7.9 \\ 5.1 & 7.8 & 5.4 & 8.4 & 8.7 \end{bmatrix}$
- (d) $\begin{bmatrix} 0.15 & 0.37 & 0.51 \\ 0.17 & 0.41 & 0.54 \\ 0.51 & 0.72 & 0.78 \\ 0.42 & 0.76 & 0.84 \\ 0.64 & 0.79 & 0.87 \end{bmatrix}$

Q (8): Determine the type of the matrix $\begin{bmatrix} -8 & 2 \\ 7 & 3 \end{bmatrix}$

- (a) unit matrix (b) column matrix
(c) row matrix (d) square matrix

Q (9): Determine the type of the matrix $\begin{bmatrix} -3 & 5 & 6 \end{bmatrix}$

- (a) unit matrix (b) column matrix
(c) row matrix (d) square matrix



Q (10): Determine the type of the matrix $\begin{bmatrix} 5 \\ -9 \\ 3 \end{bmatrix}$

- Ⓐ unit matrix Ⓑ column matrix
Ⓒ row matrix Ⓓ square matrix

Q (11): Determine the type of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Ⓐ unit matrix Ⓑ column matrix
Ⓒ row matrix Ⓓ square matrix

Q (12): Determine the type of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Ⓐ unit matrix Ⓑ column matrix
Ⓒ row matrix Ⓓ square matrix

Q (13): Determine the type of the matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

- Ⓐ unit matrix Ⓑ triangular matrix
Ⓒ zero matrix Ⓓ diagonal matrix

Q (14): Determine the type of the matrix $\begin{bmatrix} 3 & 6 & 7 \\ 0 & -3 & 5 \\ 0 & 0 & 11 \end{bmatrix}$

- Ⓐ unit matrix Ⓑ triangular matrix
Ⓒ zero matrix Ⓓ diagonal matrix

Q (15): Find the transpose of the matrix $\begin{bmatrix} -5 \\ 4 \\ 4 \end{bmatrix}$

- Ⓐ $\begin{bmatrix} 4 \\ 4 \\ -5 \end{bmatrix}$ Ⓑ $\begin{bmatrix} 4 & 4 & -5 \end{bmatrix}$
Ⓒ $\begin{bmatrix} 5 & -4 & -4 \end{bmatrix}$ Ⓓ $\begin{bmatrix} -5 & 4 & 4 \end{bmatrix}$



Q (16): Given that: $A = \begin{bmatrix} -8 & 4 & 3 \\ 4 & 1 & -1 \end{bmatrix}$, then $(A^t)^t = \dots\dots\dots$

Ⓐ $\begin{bmatrix} -8 & 4 & 4 \\ 1 & 3 & -1 \end{bmatrix}$

Ⓑ $\begin{bmatrix} -8 & 4 & 3 \\ 4 & 1 & -1 \end{bmatrix}$

Ⓒ $\begin{bmatrix} -8 & 4 \\ 3 & 4 \\ 1 & -1 \end{bmatrix}$

Ⓓ $\begin{bmatrix} -8 & 4 \\ 4 & 1 \\ 3 & -1 \end{bmatrix}$

Q (17): If X is a matrix of order 4×1 , then the order of the matrix X^T is

Ⓐ 1×1

Ⓑ 1×4

Ⓒ 4×1

Ⓓ 4×4

Q (18): Which of the following matrices is skew-symmetric?

Ⓐ $\begin{bmatrix} 10 & -9 & -2 \\ -9 & -2 & -5 \\ -2 & -5 & 4 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 0 & -3 & -5 \\ 3 & 0 & 10 \\ 5 & -10 & 0 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 3 & -5 & -2 \\ 5 & -3 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 0 & -1 & -9 \\ 1 & 0 & -6 \\ 9 & 6 & 7 \end{bmatrix}$

Q (19): Find the value of x which makes the matrix $A = \begin{bmatrix} -1 & 5x - 3 \\ -43 & -8 \end{bmatrix}$ symmetric?

Ⓐ -8

Ⓑ -43

Ⓒ -3

Ⓓ -46



Q (20): Given that $B = \begin{bmatrix} 0 & -2x & -63 \\ z + 14 & 0 & 3z \\ -3y + 6x & -12 & 0 \end{bmatrix}$ is skew-symmetric, then $x + y + z = \dots$

- Ⓐ 13
- Ⓑ 6
- Ⓒ 1
- Ⓓ 10

Q (21): Given that $B = \begin{bmatrix} -6 & -2y & -y - 3x \\ z + 5 & -9 & z - 6 \\ -9 - 2x & -11 & -4 \end{bmatrix}$ is a symmetric, what are the values of x, y, z ?

- Ⓐ $x = -9, y = 0, z = -5$
- Ⓑ $x = 9, y = 0, z = -5$
- Ⓒ $x = -6, y = 0, z = -5$
- Ⓓ $x = 9, y = 0, z = -17$



Home Work

- ① A, B and C are three cities, if the distance in kilometres between any two cities is shown in the table opposite.

Ⓐ Write a matrix represent these data.

Ⓑ Let X be the required matrix in "A" find the following:

1- What does it mean by X_{23} ?

2- What does it mean by X_{32} ?

3- what is the relation between x_{23} , x_{32} ?

	A	B	C
A	0	75	80
B	75	0	56
C	80	56	0

Ⓒ Write all elements of the second row in the matrix X.

Ⓓ Write all elements of the second column in the matrix X. What do you deduce from Ⓒ and Ⓓ?

Ⓔ Find X_{kk} when $K = 1, 2, 3$ What do you notice?

Ⓕ Complete each of the following:

1- X is a matrix of order

2- $X_{ij} = X_{ji}$ for all values

- ② What is the number of elements in each of the following matrices:

Ⓐ Matrix of order 2×3

Ⓑ Matrix of order 2×2

Ⓒ Matrix of order 3×2

- ③ Find the values of a, b, c and d if:

Ⓐ $\begin{pmatrix} 3 & -5 \\ a-3 & 3d-2 \end{pmatrix} = \begin{pmatrix} a-2 & 2b+1 \\ c & 16 \end{pmatrix}$

Ⓑ $\begin{pmatrix} 15 & 2b \\ 0 & 2a+c \end{pmatrix} = \begin{pmatrix} 3a & 10 \\ 2b-d & 10 \end{pmatrix}$



④ Find the value of each of a and b if $\begin{pmatrix} 4 & -1 \\ 2a-1 & 3b+1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3 & 7 \end{pmatrix}$

.....

⑤ If $A = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2d & -1 \\ 3e & 4 \end{pmatrix}$ where $A = B^t$
then find the value of each of d and e.

.....

⑥ If $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -3 \\ 4 & 5 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 3 & 3 \\ -4 & -5 & 5 \end{pmatrix}$

Find $A + B$, $A - B$, $A + 2B$, $B - 3A$

⑦ **Critical thinking:** If $A = (A_{xy}) \forall x, y \in \{1, 2, 3\}$ Write the matrix A, given that $A_{xy} = y - x$, then find A^t

**Lesson (1-2)****Adding and subtracting Matrices****Lesson objectives****Related Links**

-  Adding matrices.
-  Subtracting matrices.
-  Multiplying matrices.
-  Identify properties of multiplying matrices.
-  Find transpose of product of two matrices.



Scalar multiplication "Multiplying a real number by a matrix"

Example 1

If $A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -2 \\ 5 & 1 \\ 3 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 4 & 6 \end{pmatrix}$

Find each of the following if it is possible:

❶ $A + B$

❷ $C + B^t$

❸ $2A + C^t$

❹ $B + C$

Solution

❶ $A + B = \begin{pmatrix} 1 & -2 \\ 3 & 5 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ 5 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 8 & 6 \\ 7 & 4 \end{pmatrix}$

❷ $C + B^t = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 4 & 6 \end{pmatrix} + \begin{pmatrix} -1 & 5 & 3 \\ -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 2 \\ 1 & 5 & 8 \end{pmatrix}$

❸ $2A + C^t = 2 \begin{pmatrix} 1 & -2 \\ 3 & 5 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 0 & 4 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 6 & 10 \\ 8 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 0 & 4 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 14 \\ 7 & 10 \end{pmatrix}$

❹ $B + C = \begin{pmatrix} -1 & -2 \\ 5 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 \\ 3 & 4 & 6 \end{pmatrix}$ *It's impossible to find it*

**Example 2**

If $A = \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 6 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 \\ 2 & -3 \\ 4 & -8 \end{pmatrix}$ check that: $(A + B)^t = A^t + B^t$

Solution

$$\therefore A + B = \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 2 & -3 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 1 & 2 \\ 10 & -1 \end{pmatrix}$$

$$\therefore (A + B)^t = \begin{pmatrix} 2 & 1 & 10 \\ 7 & 2 & -1 \end{pmatrix} \Rightarrow \textcircled{1}$$

$$\therefore A^t + B^t = \begin{pmatrix} 2 & -1 & 6 \\ 3 & 5 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 \\ 4 & -3 & -8 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 10 \\ 7 & 2 & -1 \end{pmatrix} \Rightarrow \textcircled{2}$$

From $\textcircled{1}$ & $\textcircled{2}$: $\therefore (A + B)^t = A^t + B^t$

Example 3

Find the values of a , b , c and d that: satisfy the equation:

$$3 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2 \begin{pmatrix} a & 6 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 4 & b+4 \\ c+3 & 3 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix} = \begin{pmatrix} 2a & 12 \\ -2 & 6 \end{pmatrix} + \begin{pmatrix} 4 & b+4 \\ c+3 & 3 \end{pmatrix}$$

$$\begin{aligned} \therefore 3a &= 2a + 4 \\ \therefore 3a - 2a &= 4 \\ \therefore a &= 4 \end{aligned}$$

$$\begin{aligned} \therefore 3b &= 12 + b + 4 \\ \therefore 3b - b &= 16 \\ \therefore 2b &= 16 \\ \therefore b &= 8 \end{aligned}$$

$$\begin{aligned} \therefore 3c &= -2 + c + 3 \\ \therefore 3c - c &= 1 \\ \therefore 2c &= 1 \\ \therefore c &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \therefore 3d &= 6 + 3 \\ \therefore 3d &= 9 \\ \therefore d &= 3 \end{aligned}$$



Properties of addition of matrices

Let A , B and C be three matrices of order $m \times n$ and $\bigcirc$ is a zero matrix of the same order, then:

1- Closure property: $A + B$ form a matrix of order $m \times n$

If $A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$ is a matrix of order 2×2 , $B = \begin{pmatrix} 7 & 2 \\ -2 & 0 \end{pmatrix}$ is a matrix of order 2×2 ,

then $A + B = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 7 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ -2 & 3 \end{pmatrix}$ is a matrix of order 2×2

2- Commutative property: $A + B = B + A$

If $A = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 5 \\ -2 & 3 \end{pmatrix}$, **Show that** $A + B = B + A$

3- Associative property: $(A + B) + C = A + (B + C)$

If $A = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 5 \\ -2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$. **Show that** $(A + B) + C = A + (B + C)$

4- Additive identity property: $A + \bigcirc = \bigcirc + A = A$

for example: $\begin{pmatrix} 1 & 2 & 3 \\ 6 & -5 & 4 \\ 7 & 8 & -9 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 6 & -5 & 4 \\ 7 & 8 & -9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & -5 & 4 \\ 7 & 8 & -9 \end{pmatrix}$

5- Additive inverse property: $A + (-A) = (-A) + A = \bigcirc$

where $(-A)$ is the additive inverse of the matrix A

For example $\begin{pmatrix} 3 & 5 & 2 \\ 2 & 0 & -5 \end{pmatrix} + \begin{pmatrix} -3 & -5 & -2 \\ -2 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

where $\begin{pmatrix} -3 & -5 & -2 \\ -2 & 0 & 5 \end{pmatrix} = -\begin{pmatrix} 3 & 5 & 2 \\ 2 & 0 & -5 \end{pmatrix}$

Second: Subtraction of matrices

$$A - B + C \neq A - (B + C)$$

**Example ④**

If $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ -1 & 5 \\ 3 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 4 \\ -2 & -6 \\ 8 & -2 \end{pmatrix}$

Find the result of: $4A - 2B + \frac{1}{2}C$

Solution

$$\begin{aligned} 4A - 2B + \frac{1}{2}C &= 4 \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 4 \end{pmatrix} - 2 \begin{pmatrix} 2 & -2 \\ -1 & 5 \\ 3 & -3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 4 \\ -2 & -6 \\ 8 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 4 \\ -4 & 12 \\ 0 & 16 \end{pmatrix} - \begin{pmatrix} 4 & -4 \\ -2 & 10 \\ 6 & -6 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -1 & -3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -3 & -1 \\ -2 & 21 \end{pmatrix} \end{aligned}$$

Example ⑤

If $A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 6 & 1 \\ -1 & 5 \end{pmatrix}$

Check the following: $A - B + C \neq A - (B + C)$

Solution

$$\therefore A - B + C = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix} + \begin{pmatrix} 6 & 1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ -2 & 15 \end{pmatrix} \Rightarrow \textcircled{1}$$

$$\begin{aligned} \therefore A - (B + C) &= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} - \left[\begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix} + \begin{pmatrix} 6 & 1 \\ -1 & 5 \end{pmatrix} \right] \\ &= \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 8 & -3 \\ -4 & 11 \end{pmatrix} = \begin{pmatrix} -9 & 6 \\ 10 & -7 \end{pmatrix} \Rightarrow \textcircled{2} \end{aligned}$$

From ① & ②: $\therefore A - B + C \neq A - (B + C)$

**Example 6**

If $A = \begin{pmatrix} -1 & 1 & 4 \\ 2 & 5 & 0 \\ 3 & 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \\ 0 & -1 & -3 \end{pmatrix}$

Find the matrix X such that: $3X + 2B = A$

Solution

$$\because 3X + 2B = A$$

$$\therefore 3X = A - 2B$$

$$\therefore 3X = \begin{pmatrix} -1 & 1 & 4 \\ 2 & 5 & 0 \\ 3 & 7 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & 0 \\ 0 & -1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 - 2 \times 1 & 1 - 2 \times 2 & 4 - 2 \times 3 \\ 2 - 2 \times (-2) & 5 - 2 \times 4 & 0 - 2 \times 0 \\ 3 - 2 \times 0 & 7 - 2 \times (-1) & 2 - 2 \times (-3) \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 & -2 \\ -6 & -3 & 0 \\ 3 & 9 & 8 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} \frac{-3}{3} & \frac{-3}{3} & \frac{-2}{3} \\ \frac{-6}{3} & \frac{-3}{3} & \frac{0}{3} \\ \frac{3}{3} & \frac{9}{3} & \frac{8}{3} \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} -1 & -1 & \frac{-2}{3} \\ -2 & -1 & 0 \\ 1 & 3 & \frac{8}{3} \end{pmatrix}$$

**Example 7**

If $A = \begin{pmatrix} 2 & 3 & -2 \\ -1 & 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 & 3 \\ 5 & 2 & -4 \end{pmatrix}$

Find the matrix X that satisfies: $2 [X^t - A] = 3 B$

Solution

$$\therefore 2 [X^t - A] = 3 B$$

$$\therefore 2 X^t - 2 A = 3 B$$

$$\therefore 2 X^t = 3 B + 2 A = 3 \begin{pmatrix} 0 & -1 & 3 \\ 5 & 2 & -4 \end{pmatrix} + 2 \begin{pmatrix} 2 & 3 & -2 \\ -1 & 4 & 5 \end{pmatrix}$$

$$\therefore 2 X^t = \begin{pmatrix} 0 + 4 & -3 + 6 & 9 - 4 \\ 15 - 2 & 6 + 8 & -12 + 10 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 5 \\ 13 & 14 & -2 \end{pmatrix}$$

$$\therefore X^t = \begin{pmatrix} \frac{4}{2} & \frac{3}{2} & \frac{5}{2} \\ \frac{13}{2} & \frac{14}{2} & \frac{-2}{2} \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{5}{2} \\ \frac{13}{2} & 7 & -1 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 2 & \frac{13}{2} \\ \frac{3}{2} & 7 \\ \frac{5}{2} & -1 \end{pmatrix}$$

**Example 8**

If: $2X - 3Y = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}$ and $X + Y = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$

Find the two matrices X and Y

Solution

$$\therefore 2X - 3Y = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} \Rightarrow \textcircled{1}$$

$$\therefore X + Y = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \Rightarrow \textcircled{2} \quad (\text{By multiply by 3})$$

$$\therefore 2X - 3Y = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} \Rightarrow \textcircled{1}$$

$$\therefore 3X + 3Y = \begin{pmatrix} -3 & 6 \\ 3 & 0 \end{pmatrix} \Rightarrow \textcircled{3}$$

By adding $\textcircled{1}$ and $\textcircled{3}$

$$\therefore 5X = \begin{pmatrix} 0 & 10 \\ 5 & -5 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

From $\textcircled{2}$: $\therefore Y = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - X$

$$\therefore Y = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\therefore Y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

**PRACTICE (1)****Choose the correct answer****Evaluate:****Q (1):**

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ -3 & 4 & -2 & 1 \\ -3 & 0 & -1 & 11 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 0 & 0 \\ -1 & 5 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

Ⓐ $\begin{bmatrix} 0 & 4 & 1 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & 0 & 1 & 14 \end{bmatrix}$

Ⓑ $\begin{bmatrix} -4 & -2 & 1 & 0 \\ -4 & -1 & -2 & 1 \\ -3 & 0 & -3 & 8 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 0 & 4 & 1 & 0 \\ -4 & 9 & -2 & 1 \\ -3 & 0 & 1 & 14 \end{bmatrix}$

Ⓓ $\begin{bmatrix} -4 & -2 & 1 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & 0 & -3 & 8 \end{bmatrix}$

Q (2):

Suppose the sum $A + B + C$ makes sense. We also know that A has 2 rows and C has 3 columns. What can we say about matrix B ?

Ⓐ B is a 3×3 matrix.

Ⓑ B is a 3×2 matrix.

Ⓒ B is a 2×3 matrix.

Ⓓ B is a 2×2 matrix.

Q (3):

Given that O is the zero matrix of order 2×3 , find $\begin{pmatrix} 2 & 6 & 4 \\ 4 & 3 & -8 \end{pmatrix} + O$

Ⓐ $\begin{pmatrix} -2 & -6 & -4 \\ -4 & -3 & 8 \end{pmatrix}$

Ⓑ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Ⓒ $\begin{pmatrix} 4 & 3 & -8 \\ 2 & 6 & 4 \end{pmatrix}$

Ⓓ $\begin{pmatrix} 2 & 6 & 4 \\ 4 & 3 & -8 \end{pmatrix}$

Q (4):

Given that: $X + \begin{pmatrix} -6 & -8 \\ 6 & 5 \end{pmatrix} = O$, where O is the 2×2 zero matrix, find the value of X .

Ⓐ $\begin{pmatrix} -6 & -8 \\ 6 & 5 \end{pmatrix}$

Ⓑ $\begin{pmatrix} 6 & 8 \\ 6 & 5 \end{pmatrix}$

Ⓒ $\begin{pmatrix} 6 & 8 \\ -6 & -5 \end{pmatrix}$

Ⓓ $\begin{pmatrix} -6 & 6 \\ -8 & 5 \end{pmatrix}$



Q (5): Consider the matrix: $A = \begin{pmatrix} -1 & -1 & 5 \\ 13 & -1 & 0 \end{pmatrix}$ and if $A + B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$, find B

- Ⓐ $\begin{pmatrix} 0 & -1 & -6 \\ 13 & 0 & 2 \end{pmatrix}$
- Ⓑ $\begin{pmatrix} 2 & 1 & -6 \\ -13 & 2 & 2 \end{pmatrix}$
- Ⓒ $\begin{pmatrix} 0 & -1 & 4 \\ 13 & 0 & 2 \end{pmatrix}$
- Ⓓ $\begin{pmatrix} -2 & -1 & 6 \\ 13 & -2 & -2 \end{pmatrix}$

Q (6): Given that: $A = \begin{pmatrix} -5 & 9 & 7 \\ 10 & -2 & 0 \\ 6 & 3 & 7 \end{pmatrix}$, $(A + B)^T = \begin{pmatrix} -13 & 21 & 12 \\ 4 & 3 & 4 \\ 17 & -3 & 6 \end{pmatrix}$, find matrix B

- Ⓐ $\begin{pmatrix} -8 & 11 & 6 \\ -5 & 5 & 1 \\ 10 & -3 & -1 \end{pmatrix}$
- Ⓑ $\begin{pmatrix} -8 & -5 & 10 \\ 11 & 5 & -3 \\ 6 & 1 & -1 \end{pmatrix}$
- Ⓒ $\begin{pmatrix} -8 & -6 & 11 \\ 12 & 5 & -6 \\ 5 & 4 & -1 \end{pmatrix}$
- Ⓓ $\begin{pmatrix} -8 & 12 & 5 \\ -6 & 5 & 4 \\ 11 & -6 & -1 \end{pmatrix}$

Q (7): Given that: $B = \begin{pmatrix} 1 & -3 \\ 7 & -3 \end{pmatrix}$, $(B - B^T)^T = A$, then $a_{12} + a_{21} = \dots\dots\dots$

- Ⓐ 20
- Ⓑ 8
- Ⓒ 0
- Ⓓ -20



Q (8): Given that: $\begin{pmatrix} 3x-3 & -3 \\ -10 & y-1 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -10 & 5y-5 \end{pmatrix}$, then find x and y

- Ⓐ $x = 3, y = -1$
- Ⓑ $x = 1, y = 1$
- Ⓒ $x = -1, y = -1$
- Ⓓ $x = 1, y = -1$

Q (9): Given that: $\begin{pmatrix} 0 & -2 \\ 2a+4 & 2b+9 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & -5 \end{pmatrix}$, then find the value of a and b

- Ⓐ $a = -1, b = -7$
- Ⓑ $a = -2, b = -14$
- Ⓒ $a = -1, b = 2$
- Ⓓ $a = 2, b = -5$

Q (10): Given that: $x \begin{pmatrix} -3 \\ -8 \\ -8 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} - z \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -12 \\ -28 \\ -19 \end{pmatrix}$, then find x, y and z

- Ⓐ $x = 4, y = 2, z = -1$
- Ⓑ $x = 4, y = 2, z = -19$
- Ⓒ $x = 4, y = -2, z = 1$
- Ⓓ $x = 4, y = -2, z = 1$

Q (11): If: $x \begin{pmatrix} -4 & 6 \\ 10 & z \end{pmatrix} + y \begin{pmatrix} -7 & k \\ 0 & -5 \end{pmatrix} + 4 \begin{pmatrix} 3 & -1 \\ 10 & -2 \end{pmatrix} = 0$, then find $\begin{pmatrix} x & y \\ z & k \end{pmatrix}$

- Ⓐ $\begin{pmatrix} -4 & 4 \\ -7 & 7 \end{pmatrix}$
- Ⓑ $\begin{pmatrix} -4 & 4 \\ -16 & -16 \end{pmatrix}$
- Ⓒ $\begin{pmatrix} -4 & -14 \\ -16 & -16 \end{pmatrix}$
- Ⓓ $\begin{pmatrix} -24 & -20 \\ 3 & 1 \end{pmatrix}$

**Lesson (1-2)****Multiplying Matrices**

$$(a_{11} \quad a_{12} \quad a_{13}) \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$$

Remember that:Row $\times$ column

To multiply matrices, multiply the elements of each row in the first matrix by the elements of each column in the second matrix, then add the products.

For example, to find the product of: $A = \begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix}$

We multiply a_{11} by b_{11} , then multiply a_{12} by b_{21} and add the

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \\ ? & ? \end{pmatrix} \quad (0) \times 5 + 2 \times (-1) = -2$$

The result is the element in the first row and the first column. Repeat the same steps with the rows and columns left.

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & ? \\ ? & ? \\ ? & ? \end{pmatrix} \quad (0)(0) + (2)(1) = 2$$

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ ? & ? \\ ? & ? \end{pmatrix} \quad (-2)(5) + (-3)(-1) = -7$$

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & ? \\ ? & ? \end{pmatrix} \quad (-2)(0) + (-3)(1) = -3$$

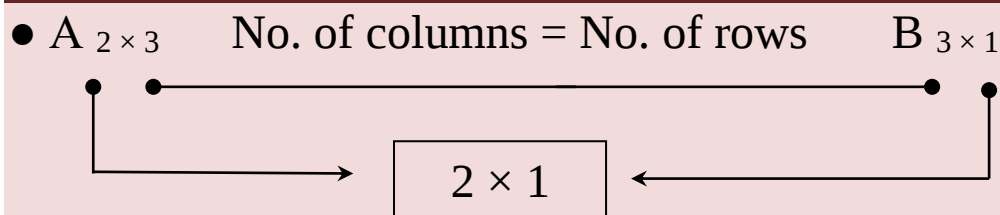
$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & -3 \\ 1 & ? \end{pmatrix} \quad (1)(0) + (4)(1) = 4$$

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & -3 \\ ? & ? \end{pmatrix} \quad (1)(5) + (4)(-1) = 1$$

$$\begin{pmatrix} 0 & 2 \\ -2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -7 & -3 \\ 1 & 4 \end{pmatrix}$$



The condition to multiply two matrices:



Example ①

Determine whether the matrix of the product AB is defined or not in each case.

(A) If the matrix A of order 3×2 and the matrix B of order 2×3 .

(B) If the matrix A of order 1×3 and the matrix B of order 1×3 .

Solution

(A) The number of columns of the matrix A equal the number of rows of the matrix B , then matrix of the product AB is defined and of order 3×3

(B) The number of columns of the matrix A is not equal to the number of rows of the matrix B , then matrix of the product AB is undefined.

Example ②

If : $A = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, find : AB and BA

Solution

$$AB = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = (2 \times -2 + -1 \times 1 + 3 \times 4) = (7)$$

$$BA = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} -2 \times 2 & -2 \times -1 & -2 \times 3 \\ 1 \times 2 & 1 \times -1 & 1 \times 3 \\ 4 \times 2 & 4 \times -1 & 4 \times 3 \end{pmatrix} = \begin{pmatrix} -4 & 2 & -6 \\ 2 & -1 & 3 \\ 8 & -4 & 12 \end{pmatrix}$$

**Example ③**

If : $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

Find: ① AB , BA is $AB = BA$? ② AC , CA is $AC = CA$?

Solution

$$AB = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 4 \times -2 + -3 \times 5 & 4 \times 1 + -3 \times 6 \\ 2 \times -2 + -1 \times 5 & 2 \times 1 + -1 \times 6 \end{pmatrix} = \begin{pmatrix} -23 & -14 \\ -9 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -2 \times 4 + 1 \times 2 & -2 \times -3 + 1 \times -1 \\ 5 \times 4 + 6 \times 2 & 5 \times -3 + 6 \times -1 \end{pmatrix} = \begin{pmatrix} -6 & 5 \\ 32 & -21 \end{pmatrix}$$

No, $AB \neq BA$ → ①

$$AC = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 4 \times -2 + -3 \times 0 & 4 \times 0 + -3 \times -2 \\ 2 \times -2 + -1 \times 0 & 2 \times 0 + -1 \times -2 \end{pmatrix} = \begin{pmatrix} -8 & 6 \\ -4 & 2 \end{pmatrix}$$

$$CA = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \times \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -2 \times 4 + 0 \times 2 & -2 \times -3 + 0 \times -1 \\ 0 \times 4 + -2 \times 2 & 0 \times -3 + -2 \times -1 \end{pmatrix} = \begin{pmatrix} -8 & 6 \\ -4 & 2 \end{pmatrix}$$

Yes, $AC = CA$ → ②

Properties of matrix multiplication

Let A , B and C be three matrices, I is the identity matrix:

① Associative property

i.e. $(A B) C = A (B C)$

② The existence of multiplicative neutral (identity) matrix property

i.e. $A I = I A = A$

③ Distributing multiplication of matrices on addition property

i.e. $A (B + C) = AB + AC$

i.e. $(A + B) C = AC + BC$

**Example 4**

If : $A = \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix}$

Check the distribution property in the following:

① $A(B + C) = AB + AC$

② $(B + C)A = BA + CA$

Solution

$$\begin{aligned} \text{① } A(B + C) &= \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix} \times \left[\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & -3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} AB + AC &= \left[\begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix} \times \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \right] + \left[\begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix} \times \begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 5 & -9 \\ -9 & 15 \end{pmatrix} + \begin{pmatrix} -4 & 8 \\ 6 & -18 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & -3 \end{pmatrix} \end{aligned}$$

$\therefore A(B + C) = AB + AC$

$$\begin{aligned} \text{② } (B + C)A &= \left[\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix} \right] \times \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -3 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA + CA &= \left[\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix} \right] + \left[\begin{pmatrix} -2 & 6 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} 18 & -6 \\ -7 & 2 \end{pmatrix} + \begin{pmatrix} -20 & 4 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -3 & 0 \end{pmatrix} \end{aligned}$$

$\therefore (B + C)A = BA + CA$



The identity matrix (unit matrix)

$$I_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark

$$(A B)^t = B^t A^t$$

Example 7

If : $A = \begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$, prove that : $(AB)^2 \neq A^2 B^2$

Solution

$$AB = A \times B = \begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ 12 & -10 \end{pmatrix}$$

$$(AB)^2 = AB \times AB = \begin{pmatrix} -3 & 5 \\ 12 & -10 \end{pmatrix} \times \begin{pmatrix} -3 & 5 \\ 12 & -10 \end{pmatrix} = \begin{pmatrix} 69 & -65 \\ -156 & 160 \end{pmatrix}$$

$$A^2 = A \times A = \begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix} \times \begin{pmatrix} -1 & 3 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 13 & 4 \\ -12 & 16 \end{pmatrix}$$

$$B^2 = B \times B = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -8 \\ 0 & 1 \end{pmatrix}$$

$$A^2 B^2 = A^2 \times B^2 = \begin{pmatrix} 13 & 4 \\ -12 & 16 \end{pmatrix} \times \begin{pmatrix} 9 & -8 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 117 & -100 \\ -108 & 112 \end{pmatrix}$$

$$\therefore (AB)^2 \neq A^2 B^2$$

**Example 8**

If : $A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 1 \\ -3 & 9 \\ 2 & -8 \end{pmatrix}$, Check the relation: $(AB)^t = B^t A^t$

Solution

$$AB = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 5 & 7 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ -3 & 9 \\ 2 & -8 \end{pmatrix} = \begin{pmatrix} 23 & -39 \\ 17 & -8 \end{pmatrix}$$

$$(AB)^t = \begin{pmatrix} 23 & 17 \\ -39 & -8 \end{pmatrix} \rightarrow \textcircled{1}$$

$$B^t A^t = \begin{pmatrix} 6 & -3 & 2 \\ 1 & 9 & -8 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 5 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 23 & 17 \\ -39 & -8 \end{pmatrix} \rightarrow \textcircled{2}$$

From ① & ②: $\therefore (AB)^t = B^t A^t$

Example 9

Find the matrix X if: $\begin{pmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{pmatrix} \times X = \begin{pmatrix} -1 \\ 3 \\ 13 \end{pmatrix}$

Solution

Of order $3 \times \underline{2}$

Of order $\underline{2} \times 1$

Of order 3×1

Let $X = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 4 \end{pmatrix} \times \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 13 \end{pmatrix} \quad \therefore \begin{pmatrix} 2a - b \\ 3a \\ a + 4b \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 13 \end{pmatrix}$$

$$\therefore 3a = 3 \quad \Rightarrow a = 1 \quad \therefore a + 4b = 13 \quad \Rightarrow 1 + 4b = 13 \quad \Rightarrow b = 3$$

$$\therefore X = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

**PRACTICE (1)****Choose the correct answer**

Q (1): Given that: $A = \begin{pmatrix} -5 & -5 \\ -6 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 6 \\ -3 & 5 \end{pmatrix}$, find $(A + B) A$

- Ⓐ $\begin{pmatrix} -1 & -21 \\ 11 & 111 \end{pmatrix}$
- Ⓑ $\begin{pmatrix} -1 & 11 \\ -21 & 111 \end{pmatrix}$
- Ⓒ $\begin{pmatrix} 59 & -71 \\ -49 & 61 \end{pmatrix}$
- Ⓓ $\begin{pmatrix} -6 & -4 \\ -15 & 17 \end{pmatrix}$

Q (2): Given that: $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$, then $A^2 = \dots\dots\dots$

- Ⓐ $\begin{pmatrix} 9 & 4 \\ 1 & 1 \end{pmatrix}$
- Ⓑ $\begin{pmatrix} 9 & 4 \\ -1 & 1 \end{pmatrix}$
- Ⓒ $\begin{pmatrix} 7 & 8 \\ -4 & -1 \end{pmatrix}$
- Ⓓ $\begin{pmatrix} 7 & 8 \\ 4 & 1 \end{pmatrix}$

Q (3): If: $A = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ and $B = (5 \ 0)$, then $(BA)^T = \dots\dots\dots$

- Ⓐ $\begin{pmatrix} 30 & 42 \\ 0 & 14 \end{pmatrix}$
- Ⓑ (30)
- Ⓒ (-30)
- Ⓓ $\begin{pmatrix} 30 & 0 \\ 42 & 14 \end{pmatrix}$



Q (4): If: $BA = \begin{pmatrix} 8 & -4 \\ -7 & -5 \end{pmatrix}$, then $A^T B^T = \dots\dots\dots$

Ⓐ $\begin{pmatrix} -5 & -7 \\ -4 & 8 \end{pmatrix}$

Ⓑ $\begin{pmatrix} 8 & -7 \\ -4 & -5 \end{pmatrix}$

Ⓒ $\begin{pmatrix} 8 & -4 \\ -7 & -5 \end{pmatrix}$

Ⓓ $\begin{pmatrix} -5 & -4 \\ -7 & 8 \end{pmatrix}$

Q (5): Consider the matrix: $A = \begin{pmatrix} -5 & -6 \\ 5 & 0 \end{pmatrix}$, find $A^2 + 5A + 30I$

Ⓐ $\begin{pmatrix} 0 & 30 \\ 30 & 0 \end{pmatrix}$

Ⓑ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Ⓒ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Ⓓ $\begin{pmatrix} 66 & -55 \\ 0 & 55 \end{pmatrix}$

Q (6): Let: $X = (-1 \quad -1 \quad 1)$, $Y = (0 \quad 1 \quad 2)$, find $X^T Y$ and $X Y^T$

Ⓐ $\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix}, 1$

Ⓑ $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}, -1$

Ⓒ $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix}, 1$

Ⓓ $\begin{pmatrix} 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}, 1$

Q (7):

If: X of order 2×3 and Y of order 1×3 , then which of the following is possible

- (a) $\mathbf{X} \mathbf{Y}$
 (b) $\mathbf{X}^T \mathbf{Y}$
 (c) $\mathbf{Y} \mathbf{X}$
 (d) $\mathbf{X} \mathbf{Y}^T$

Q (8):

If: X of order 3×2 and Y of order 1×3 , then YX is of order

- (a) 3×2
 (b) 1×3
 (c) 2×1
 (d) 1×2

Q (9):

For: $A = \begin{bmatrix} 4 & -5 \\ 4 & -5 \end{bmatrix}$, then $A^2 = \dots\dots\dots$

- (a) $-A$
 (b) $-4A$
 (c) $4A$
 (d) $2A$

Q (10):

Consider: $X = \begin{bmatrix} -3 & -3 \\ 5 & -6 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 3 \\ 6 & -6 \end{bmatrix}$, then $X^2 - Y^2 = \dots\dots\dots$

- (a) $\begin{bmatrix} -25 & 42 \\ -15 & -33 \end{bmatrix}$
 (b) $\begin{bmatrix} -25 & -15 \\ 42 & -33 \end{bmatrix}$
 (c) $\begin{bmatrix} -25 & 57 \\ -30 & -33 \end{bmatrix}$
 (d) $\begin{bmatrix} -25 & -30 \\ 57 & -33 \end{bmatrix}$



Home Work (1)

- ① If $A = \begin{pmatrix} -2 & 0 & -1 \\ 4 & 5 & 0 \end{pmatrix}$ and $K_1 = 2$, $K_2 = -1$ then find each of the following matrices:
 $K_1 A$ and $K_2 A$
- ② If $A = \begin{pmatrix} -7 & 0 & -5 \\ 4 & 7 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -8 & 4 \\ 0 & 7 \\ -6 & -5 \end{pmatrix}$ then find the result of the following operations if "possible", give reasons in case of impossible solution.
- (A) $A + B$
(B) $A + B^t$
(C) $A^t + B$
- ③ If $X = \begin{pmatrix} -4 & -2 \\ 3 & 6 \\ 0 & 4 \end{pmatrix}$, $Y = \begin{pmatrix} 5 & 1 \\ 0 & -2 \\ 4 & -3 \end{pmatrix}$, $Z = \begin{pmatrix} -2 & -4 \\ -3 & 2 \\ 6 & 0 \end{pmatrix}$ then find the matrix $3X - Y + Z$
- ④ If $A = \begin{pmatrix} 4 & 8 & -6 \\ 2 & -4 & 8 \\ 6 & 12 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -6 & 2 \\ 4 & -10 & 0 \\ -1 & 8 & -4 \end{pmatrix}$, then find the matrix X where $X = 2A - 3B$

Home Work (2)

- ① If $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 \\ 3 & 9 \end{pmatrix}$, then find each of the following:
- (A) AB
(B) BA
(C) $(A + B)A$
- ② If $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x & 7 \\ 3 & y \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 11 & 18 \end{pmatrix}$ find the value of each of x, y:
- ③ **Critical thinking:** If A and B are two matrices and, $\bigcirc$ is the zero matrix, $AB = \bigcirc$ Does it always mean that $A = \bigcirc$ or $B = \bigcirc$. Take $A = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$, then show your opinion.

**Lesson (1-4)****Determinates****Lesson objectives****Related Links**

- ✚ Identify Determinant of square matrix of the second order.
- ✚ Identify Determinant of the square matrix of the third order..
- ✚ Identify Determinant of the triangular matrix.
- ✚ Find the area of the triangle using the determinants.
- ✚ Solve the system of linear equations using the determinants.

**Determinants**

If A is a square matrix of order 2×2 where:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the determinant of the matrix A is denoted by $|A|$ and is called determinant of the second order and is the number defined as follows:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a d - b c$$

Other diagonal
Principle diagonal

Example ①

Find: (A) $\begin{vmatrix} 2 & -2 \\ 3 & 6 \end{vmatrix}$ (B) $\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}$ (C) $\begin{vmatrix} -4 & 1 \\ 1 & -3 \end{vmatrix}$

Solution

$$(A) \begin{vmatrix} 2 & -2 \\ 3 & 6 \end{vmatrix} = (2 \times 6) - (-2 \times 3) = 18$$

$$(B) \begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix} = (3 \times -2) - (0 \times 2) = -6$$

$$(C) \begin{vmatrix} -4 & 1 \\ 1 & -3 \end{vmatrix} = (-4 \times -3) - (1 \times 1) = 11$$



Third order determinant

The determinant of the matrix of order 3×3 is called a third order determinant. To find the value

of the third order determinant, $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ then:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Example 2

Find the value of each of the following determinants:

$$(A) \begin{vmatrix} -1 & 2 & 1 \\ 2 & -2 & 3 \\ 3 & 0 & 1 \end{vmatrix}$$

$$(B) \begin{vmatrix} 2 & 1 & 2 \\ -2 & 0 & 2 \\ -1 & 3 & 0 \end{vmatrix}$$

$$(C) \begin{vmatrix} 2 & -2 & -3 \\ 4 & 3 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$

Solution

$$\begin{aligned} (A) \begin{vmatrix} -1 & 2 & 1 \\ 2 & -2 & 3 \\ 3 & 0 & 1 \end{vmatrix} &= (-1) \begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix} - (2) \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + (1) \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} \\ &= (-1 \times -2) - (2 \times -7) + (1 \times 6) = -6 \end{aligned}$$

$$\begin{aligned} (B) \begin{vmatrix} 2 & 1 & 2 \\ -2 & 0 & 2 \\ -1 & 3 & 0 \end{vmatrix} &= (2) \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} - (1) \begin{vmatrix} -2 & 2 \\ -1 & 0 \end{vmatrix} + (2) \begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix} \\ &= (2 \times -6) - (1 \times 2) + (2 \times -6) = -26 \end{aligned}$$

$$\begin{aligned} (C) \begin{vmatrix} 2 & -2 & -3 \\ 4 & 3 & -2 \\ 5 & 2 & -3 \end{vmatrix} &= (2) \begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix} - (-2) \begin{vmatrix} 4 & -2 \\ 5 & -3 \end{vmatrix} + (-3) \begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix} \\ &= (2 \times -5) - (-2 \times -2) + (-3 \times -7) = 7 \end{aligned}$$

**Important Notes**

1- If A is a square matrix of order 3×3 in the form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ and the determinant A is denoted by the symbol } |A| \text{ where:}$$

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}|a_{11}| - a_{12}|a_{12}| + a_{13}|a_{13}| \end{aligned}$$

2- Notice that, we multiply each element in the minor determinant corresponding to it preceeding by the signs +, -, +, ... respectively and the sign of the minor determinant corresponding to the element, a_{ij} is determined by the rule:

Sign of $|a_{ij}|$ is the same as the sign of $(-1)^{i+j}$

For example: the sign of $|a_{12}|$ is the same as the sign of $(-1)^{1+2}$ which is negative

The sign of $|a_{13}|$ is the same as the sign of $(-1)^{1+3}$ which is positive

To determine the sign of each minor determinant corresponding to an element, we add the two orders of row and column which intersect at the element:

➤ If the sum of the two orders is **even**, then the sign is **positive**.

➤ If the sum of the two orders is **odd**, then the sign is **negative**.

We notice that, the rule of signs of the minor determinant is as follows: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Example 3

Find the value of the determinants: $\begin{vmatrix} -1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} -1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{vmatrix} &= (-1) \begin{vmatrix} 3 & 3 \\ 0 & 2 \end{vmatrix} - (2) \begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix} + (3) \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix} \\ &= (-1 \times 6) - (2 \times 0) + (3 \times 0) = -6 \end{aligned}$$

This matrix is called **triangular matrix**

(We can obtain the result by multiply the main diagonal $(-1 \times 3 \times 2 = -6)$)



Finding area of a triangle by using Determinants

You can use the determinant to find the surface area of the triangle in terms of the coordinates of the vertices of the triangle as follows:

Area of the triangle in which its vertices are: X (a, b), Y (c, d), Z (e, f) equals $|A|$ where:

$$A = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

Remember

$|A|$ means the positive value of A.

Example 4

Find the area of the triangle ABC in which A (-2, -2), B (3, 1), C (-4, 3)

Solution

$$\begin{aligned} A &= \frac{1}{2} \begin{vmatrix} -2 & -2 & 1 \\ 3 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \frac{1}{2} \left[(-2) \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 3 & 1 \\ -4 & 1 \end{vmatrix} + (1) \begin{vmatrix} 3 & 1 \\ -4 & 3 \end{vmatrix} \right] \\ &= \frac{1}{2} [(-2 \times -2) - (-2 \times 7) + (1 \times 13)] = 15.5 \text{ square units} \end{aligned}$$

Solving a system of linear equations by Cramer's rule

1- Solving a system of Linear equations in two variables

If we have a system of linear equations in two variables as follows:

$$a x + b y = m$$

$$c x + d y = n$$

$$x = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{m d - n b}{a d - c b}, \quad y = \frac{\Delta_y}{\Delta} = \frac{\begin{vmatrix} a & m \\ b & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{a n - b m}{a d - c b}$$

**Example 5**

Solve the system of the following equations using Cramer's rule:

$$x + 2y = 0$$

&

$$2x - 3y = 1$$

Solution

$$\Delta = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = (1 \times -3) - (2 \times 2) = -7$$

$$\Delta x = \begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} = (0 \times -3) - (2 \times 1) = -5$$

$$\Delta y = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = (1 \times 1) - (0 \times 2) = 1$$

$$x = \frac{\Delta x}{\Delta} = \frac{-5}{-7} = \frac{5}{7}, \quad y = \frac{\Delta y}{\Delta} = \frac{1}{-7} = -\frac{1}{7} \quad \text{The S.S.} = \left\{ \left(\frac{5}{7}, -\frac{1}{7} \right) \right\}$$

Example 6

Solve the system of the following equations using Cramer's rule:

$$2x + 3y = -2$$

&

$$-2x + 4y = 3$$

Solution

$$\Delta = \begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix} = (2 \times 4) - (3 \times -2) = 14$$

$$\Delta x = \begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix} = (-2 \times 4) - (3 \times 3) = -17$$

$$\Delta y = \begin{vmatrix} 2 & -2 \\ -2 & 3 \end{vmatrix} = (2 \times 3) - (-2 \times -2) = 2$$

$$x = \frac{\Delta x}{\Delta} = \frac{-17}{14} = -\frac{17}{14}, \quad y = \frac{\Delta y}{\Delta} = \frac{2}{14} = \frac{1}{7} \quad \text{The S.S.} = \left\{ \left(-\frac{17}{14}, \frac{1}{7} \right) \right\}$$



2- Solving systems of Linear equations in three variables

If we have a system of linear equations in three variables as follows:

$$a_1 x + b_1 y + c_1 z = m$$

$$a_2 x + b_2 y + c_2 z = n$$

$$a_3 x + b_3 y + c_3 z = k$$

Then, by a similar way as we did in case of system of linear equations in two variables:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{determinant of the coefficients}$$

$$\Delta x = \begin{vmatrix} m & b_1 & c_1 \\ n & b_2 & c_2 \\ k & b_3 & c_3 \end{vmatrix} = \text{determinant of the variable } x$$

we get it by changing the elements of the first column (coefficients of x) by the constants m, n, k

$$\Delta y = \begin{vmatrix} a_1 & m & c_1 \\ a_2 & n & c_2 \\ a_3 & k & c_3 \end{vmatrix} = \text{determinant of the variable } y$$

we get it by changing the elements of the second column (coefficients of y) by the constants m, n, k

$$\Delta z = \begin{vmatrix} a_1 & b_1 & m \\ a_2 & b_2 & n \\ a_3 & b_3 & k \end{vmatrix} = \text{determinant of the variable } z$$

we get it by changing the elements of the third column (coefficients of z) by the constants m, n, k

$$\text{Now, If } \Delta \neq \text{zero, then: } x = \frac{\Delta x}{\Delta}, y = \frac{\Delta y}{\Delta}, z = \frac{\Delta z}{\Delta}$$

**Example 7**

Solve the system of the following linear equations using Cramer's rule:

$$2x - 2y + 3z = 4, \quad 2x - 3y + z = 3, \quad 4x - 3y + z = 0$$

Solution

$$\Delta = \begin{vmatrix} 2 & -2 & 3 \\ 2 & -3 & 1 \\ 4 & -3 & 1 \end{vmatrix} = 2(-3 + 3) - (-2)(2 - 4) + 3(-6 + 12) = 14$$

$$\Delta x = \begin{vmatrix} 4 & -2 & 3 \\ 3 & -3 & 1 \\ 0 & -3 & 1 \end{vmatrix} = 4(-3 + 3) - (-2)(3 - 0) + 3(-9 - 0) = -21$$

$$\Delta y = \begin{vmatrix} 2 & 4 & 3 \\ 2 & 3 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 2(3 - 0) - 4(2 - 4) + 3(0 - 12) = -22$$

$$\Delta z = \begin{vmatrix} 2 & -2 & 4 \\ 2 & -3 & 3 \\ 4 & -3 & 0 \end{vmatrix} = 2(0 + 9) - (-2)(0 - 12) + 4(-6 + 12) = 18$$

$$x = \frac{\Delta x}{\Delta} = \frac{-21}{14} = -\frac{3}{2}$$

$$y = \frac{\Delta y}{\Delta} = \frac{-22}{14} = -\frac{11}{7}$$

$$z = \frac{\Delta z}{\Delta} = \frac{18}{14} = \frac{9}{7}$$

$$\text{The S.S.} = \left\{ \left(-\frac{3}{2}, -\frac{11}{7}, \frac{9}{7} \right) \right\}$$

**PRACTICE****Choose the correct answer**

Q (1): Find all the values of x for which $\begin{vmatrix} x & -2 \\ -2 & x \end{vmatrix} + \begin{vmatrix} 6 & 3 \\ 1 & 8 \end{vmatrix} = 45$

- Ⓐ 2 or -2
- Ⓑ 4 or -4
- Ⓒ 1 or -1
- Ⓓ 2 or -4

Q (2): Find all the values of x for which $\begin{vmatrix} x & -5 \\ -5 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 4 & 8 \end{vmatrix} = 40$

- Ⓐ 7 or -7
- Ⓑ 49 or -49
- Ⓒ 8 or -8
- Ⓓ 64 or -64

Q (3): Solve: $\begin{vmatrix} x & 5 \\ 4 & -9 \end{vmatrix} = 16$

- Ⓐ $x = -\frac{20}{9}$
- Ⓑ $x = -4$
- Ⓒ $x = 36$
- Ⓓ $x = \frac{4}{9}$

Q (4): Find the value of: $\begin{vmatrix} a+x & 5a \\ b+7y & -7b \end{vmatrix}$

- Ⓐ $-7bx - 35ay$
- Ⓑ $-7bx - 35ay - 2ab$
- Ⓒ $-7bx - 35ay - 12ab$
- Ⓓ $-7bx + 35ay - 2ab$



Q (5): Find the value of the determinant: $\begin{vmatrix} \cot \theta & \tan \theta \\ -\cot \theta & \tan \theta \end{vmatrix}$

- Ⓐ 0
- Ⓑ 1
- Ⓒ -2
- Ⓓ 2

Q (6): Solve: $\begin{vmatrix} x & 5 \\ 4 & -9 \end{vmatrix} = 16$

- Ⓐ $x = -\frac{20}{9}$
- Ⓑ $x = -4$
- Ⓒ $x = 36$
- Ⓓ $x = \frac{4}{9}$

Q (7): Find the value of: $\begin{vmatrix} -2 & 5 & -8 \\ 5 & 1 & 8 \\ 0 & 0 & -5 \end{vmatrix}$

- Ⓐ 139
- Ⓑ -191
- Ⓒ 135
- Ⓓ -115

Q (8): Consider the determinant $\begin{vmatrix} x & y & z \\ y & x & z \\ z & y & x \end{vmatrix}$ given that: $x^3 + y^3 + z^3 = -73$ and $xyz = -8$, then find the value of the determinant

- Ⓐ -56
- Ⓑ -49
- Ⓒ -97
- Ⓓ -81



Q (9): Use the properties of determinants to evaluate $\begin{vmatrix} -1 & 0 & 0 \\ -5 & 5 & 0 \\ 9 & -4 & -4 \end{vmatrix}$

- (a) 0
- (b) 180
- (c) 20
- (d) 25

Q (10): William is solving simultaneous equations using Cramer's rule. Where:
 $\Delta_x = \begin{vmatrix} -9 & -8 \\ 2 & -2 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} 9 & -9 \\ -3 & 2 \end{vmatrix}$
 What is the value of Δ ?

- (a) $\begin{vmatrix} 9 & -9 \\ -3 & 2 \end{vmatrix}$
- (b) $\begin{vmatrix} -9 & 9 \\ 2 & -3 \end{vmatrix}$
- (c) $\begin{vmatrix} 9 & -8 \\ -3 & -2 \end{vmatrix}$
- (d) $\begin{vmatrix} -9 & -8 \\ 2 & -2 \end{vmatrix}$

Q (11): Use determinants to solve the system
 $-8x - 4y = -8,$
 $9x - 6y = -9.$

- (a) $x = 6, y = -10$
- (b) $x = \frac{1}{7}, y = \frac{12}{7}$
- (c) $x = 1, y = 12$
- (d) $x = -\frac{1}{7}, y = -\frac{12}{7}$

Q (12): Use determinate to find the area of triangle whose vertices are (0 ,2) , (3, 5) and (-3 , 2)

- (a) 18
- (b) 9
- (c) 4.5
- (d) 8



Q (13):

Use determinants to solve the system

$$-7x + 2y + 5 = 0,$$

$$x + 8y - 1 = 0.$$

Ⓐ $x = \frac{1}{29}, y = \frac{21}{29}$

Ⓑ $x = -\frac{21}{29}, y = -\frac{1}{29}$

Ⓒ $x = \frac{7}{9}, y = \frac{1}{27}$

Ⓓ $x = \frac{21}{29}, y = \frac{1}{29}$

Q (14):

Use determinants to solve the system

$$5x = -2y - 5 + 3z,$$

$$-3x - y + 1 = 2z,$$

$$2y - z = -5x + 3.$$

Ⓐ $x = 4, y = 3, z = \frac{31}{3}$

Ⓑ $x = 21, y = -56, z = -4$

Ⓒ $x = -21, y = 56, z = 4$

Ⓓ $x = -27, y = 29, z = -26$

Q (15):

Use determinants to solve the system

$$\begin{vmatrix} -1 & z \\ -4 & y \end{vmatrix} = 23,$$

$$\begin{vmatrix} 2 & y \\ -5 & x \end{vmatrix} = 13,$$

$$\begin{vmatrix} 3 & x \\ 5 & z \end{vmatrix} = 51.$$

Ⓐ $x = -6, y = 5, z = 7$

Ⓑ $x = -7, y = 5, z = 9$

Ⓒ $x = -6, y = 7, z = 6$

Ⓓ $x = 7, y = -5, z = 6$



Home Work

① Find: (A) $\begin{vmatrix} -4 & 2 \\ -1 & 3 \end{vmatrix}$ (B) $\begin{vmatrix} 2 & 3 \\ 1 & -5 \end{vmatrix}$ (C) $\begin{vmatrix} 3 & -2 \\ -4 & 6 \end{vmatrix}$

② Find the value of each of the following determinants:

(A) $\begin{vmatrix} 2 & 1 & 3 \\ 3 & 5 & -3 \\ -2 & 0 & 4 \end{vmatrix}$ (B) $\begin{vmatrix} -2 & 3 & 4 \\ 0 & 6 & 6 \\ -3 & 4 & -2 \end{vmatrix}$ (C) $\begin{vmatrix} -1 & -2 & 3 \\ 3 & 0 & 4 \\ 4 & 0 & -2 \end{vmatrix}$

③ Find the area of the triangle ABC in which A (-3, -3), B (4, 2), C (-3, 2)

④ Find the area of the triangle ABC in which A (-2, 0), B (-3, 4), C (3, 0)

⑤ Solve the system of the following equations using Cramer's rule:

$$2x + 3y = 2 \quad \& \quad -3x + 4y = 5$$

⑥ Solve the system of the following equations using Cramer's rule:

$$6x = 5y - 3 \quad \& \quad 3y - 4 = 2x$$

⑦ Solve the system of the following equations using Cramer's rule:

$$2x - 3y + 2z = 4 \quad , \quad 3x - 2y - 4z = 5 \quad , \quad 4x + 3y + z = 6$$

⑧ Solve the system of the following equations using Cramer's rule:

$$2x - 3y + 5z = 7 \quad , \quad 3x + 4y + 2z = 11 \quad , \quad x - 2y + 7z = 16$$

⑨ Solve the system of the following equations using Cramer's rule:

$$2x + y - z = -1 \quad , \quad 2x + y + 4z = 1 \quad , \quad 5x - 3y + 2z = 3$$

**Lesson (1-5)****Multiplicative inverse of Matrix****Lesson objectives****Related Links**

- Find the multiplicative inverse of a matrix of order 2×2 .
- Solve a system of two linear equations using the inverse matrix.

The multiplicative inverse of a 2×2 matrix:

If the matrix A has a multiplicative inverse,
then we denoted it by the symbol A^{-1}
where: $AA^{-1} = A^{-1}A = I$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the multiplicative inverse of
the matrix A existed when the determinant of A equals $\Delta \neq 0$

let the matrix A^{-1} be the multiplicative inverse of the matrix A , and the
determinant of A equals $\Delta \neq 0$ then:

$$A^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example 1

If $A = \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$. **Prove that** for the matrix A , there is a multiplicative inverse,
then find it.

Solution

$$\text{Determinant of } A = \begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = -6 - 20 = -26$$

$$\therefore \Delta \neq 0$$

$\therefore$ Matrix A has a multiplicative inverse

$$\therefore A^{-1} = \frac{1}{-26} \begin{pmatrix} 3 & -4 \\ -5 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-3}{26} & \frac{2}{13} \\ \frac{5}{26} & \frac{1}{13} \end{pmatrix}$$

**Example 2**

If $A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$. **Prove that** for the matrix A , there is a multiplicative inverse, then find it.

Solution

$$\text{Determinant of } A = \begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} = 8 - 0 = 8$$

$$\therefore \Delta \neq 0$$

$\therefore$ Matrix A has a multiplicative inverse

$$\therefore A^{-1} = \frac{1}{8} \begin{pmatrix} 4 & 0 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{3}{8} & \frac{3}{8} \end{pmatrix}$$

Example 3

Find the values of b which make the matrix $A = \begin{pmatrix} b & 3 \\ 12 & b \end{pmatrix}$ have a multiplicative inverse.

Solution

$\therefore$ Matrix A has a multiplicative inverse

$$\therefore \Delta \neq 0$$

$$\text{Put } \begin{vmatrix} b & 3 \\ 12 & b \end{vmatrix} = 0$$

$$\therefore b^2 - 36 = 0$$

$$\therefore b^2 = 36$$

$$\therefore b = \pm\sqrt{36} = \pm 6$$

$\therefore$ When $b \in \mathbb{R} - \{-6, 6\}$, there is a multiplicative inverse for the given matrix.

**Example 4**

Find the values of x which make the matrix $B = \begin{pmatrix} x & -8 \\ -2 & x \end{pmatrix}$ has no multiplicative inverse.

Solution

$\because$ Matrix A has a multiplicative inverse

$$\begin{aligned} \therefore \Delta &= 0 & \therefore \begin{vmatrix} x & -8 \\ -2 & x \end{vmatrix} &= 0 & \therefore x^2 - 16 &= 0 \\ \therefore x^2 &= 16 & \therefore x &= \pm\sqrt{16} = \pm 4 & \therefore x &\in \{-4, 4\} \end{aligned}$$

Example 5

If $X = \begin{pmatrix} -1 & 0 \\ x & 1 \end{pmatrix}$. **Prove that** $X^{-1} = X$

Solution

$$\begin{aligned} \therefore \Delta &= \begin{vmatrix} -1 & 0 \\ x & 1 \end{vmatrix} = -1 \neq 0 \\ \therefore X^{-1} &= \frac{1}{-1} \begin{pmatrix} 1 & 0 \\ -x & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ x & 1 \end{pmatrix} = X \end{aligned}$$

Example 6

If : $B = \begin{pmatrix} x & -xy \\ 0 & y \end{pmatrix}$, **then Find:** B^{-1} given that $xy \neq 0$

Solution

$$\begin{aligned} \therefore \Delta &= \begin{vmatrix} x & -xy \\ 0 & y \end{vmatrix} = xy - 0 = xy \\ \therefore B^{-1} &= \frac{1}{xy} \begin{pmatrix} y & xy \\ 0 & x \end{pmatrix} = \begin{pmatrix} \frac{1}{x} & 1 \\ 0 & \frac{1}{y} \end{pmatrix} \end{aligned}$$

**Example 7**

Solve the system of the following linear equations using matrices:

$$2x - 3y = 4 \quad , \quad 3x + 4y = 23$$

Solution

First: Cramer's rule: $\Delta = \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = 17$

$$\Delta x = \begin{vmatrix} 4 & -3 \\ 23 & 4 \end{vmatrix} = 85$$

$$\Delta y = \begin{vmatrix} 2 & 4 \\ 3 & 23 \end{vmatrix} = 34$$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{85}{17} = 5 \quad \& \quad \therefore y = \frac{\Delta y}{\Delta} = \frac{34}{17} = 2$$

Second: Matrix: Let the matrix equation be $AX = C$

$$\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 23 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 23 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 4 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 23 \end{pmatrix} = \begin{pmatrix} \frac{4}{17} & \frac{3}{17} \\ \frac{-3}{17} & \frac{2}{17} \end{pmatrix} \begin{pmatrix} 4 \\ 23 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\therefore x = 5 \quad \& \quad y = 2$$



PRACTICE

Choose the correct answer

Q (1): Find the inverse of the matrix $A = \begin{pmatrix} 3 & 3 \\ 1 & 6 \end{pmatrix}$

Ⓐ $A^{-1} = \frac{1}{21} \begin{pmatrix} -3 & 1 \\ 3 & -6 \end{pmatrix}$

Ⓑ $A^{-1} = \frac{1}{15} \begin{pmatrix} 6 & -3 \\ -1 & 3 \end{pmatrix}$

Ⓒ $A^{-1} = \frac{1}{15} \begin{pmatrix} -3 & 1 \\ 3 & -6 \end{pmatrix}$

Ⓓ $A^{-1} = \frac{1}{15} \begin{pmatrix} 6 & -3 \\ -1 & 3 \end{pmatrix}$

Q (2): Find the set of real values of A for which $A = \begin{pmatrix} a & 25 \\ 1 & a \end{pmatrix}$ has a multiplicative inverse.

Ⓐ $\mathbb{R}$

Ⓑ $\mathbb{R} - \{-5, 5\}$

Ⓒ $\{-5, 5\}$

Ⓓ $\mathbb{R} - [-5, 5]$

Q (3): Given: $A = \begin{pmatrix} x^3 & -x^3y^3 \\ 0 & y^3 \end{pmatrix}$ find A^{-1}

Ⓐ $\begin{pmatrix} \frac{1}{x^3} & -1 \\ 0 & \frac{1}{y^3} \end{pmatrix}$

Ⓑ $\begin{pmatrix} \frac{1}{x^3} & 1 \\ 0 & \frac{1}{y^3} \end{pmatrix}$

Ⓒ $\begin{pmatrix} \frac{1}{x^3} & 0 \\ 1 & \frac{1}{y^3} \end{pmatrix}$

Ⓓ $\begin{pmatrix} -\frac{1}{x^3} & 0 \\ 1 & -\frac{1}{y^3} \end{pmatrix}$



Q (4): Find the multiplicative inverse of : $\begin{pmatrix} 69 & 0 \\ 0 & 69 \end{pmatrix}$

- Ⓐ $\begin{pmatrix} \frac{1}{69} & 0 \\ 0 & \frac{1}{69} \end{pmatrix}$
- Ⓑ $\begin{pmatrix} -\frac{1}{69} & 0 \\ 0 & -\frac{1}{69} \end{pmatrix}$
- Ⓒ $\begin{pmatrix} -69 & 0 \\ 0 & -69 \end{pmatrix}$
- Ⓓ $\begin{pmatrix} 69 & 0 \\ 0 & -69 \end{pmatrix}$

Q (5): Given: $B = \begin{pmatrix} -2 & -5 \\ -6 & -10 \end{pmatrix}$ find $A \times B = I$, the find matrix A

- Ⓐ $\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}$
- Ⓑ $\begin{pmatrix} -10 & -5 \\ -6 & -2 \end{pmatrix}$
- Ⓒ $\begin{pmatrix} -10 & 5 \\ 6 & -2 \end{pmatrix}$
- Ⓓ $\begin{pmatrix} \frac{1}{5} & \frac{1}{2} \\ \frac{3}{5} & 1 \end{pmatrix}$



Home Work

- ① Show the matrices which have inverses, and the matrices which have not inverses in the following. if there is an inverse find it.

A $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

B $\begin{pmatrix} 2 & 6 \\ -1 & 3 \end{pmatrix}$

C $\begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix}$

D $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

E $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$

F $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

G $\begin{pmatrix} 3 & 9 \\ 2 & 6 \end{pmatrix}$

H $\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$

- ② What are the values of a which make to each of the following matrices a multiplicative inverse.

A $\begin{pmatrix} a & 1 \\ 6 & 3 \end{pmatrix}$

B $\begin{pmatrix} a & 9 \\ 4 & a \end{pmatrix}$

C $\begin{pmatrix} a & 4 \\ 2 & a-2 \end{pmatrix}$

D $\begin{pmatrix} a-1 & -2 \\ 1 & a-1 \end{pmatrix}$

- ③ If $X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ then prove that $X^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

- ④ Find the matrix A if: $A \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- ⑤ If $X = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$, Prove that $(XY)^{-1} = Y^{-1}X^{-1}$

- ⑥ Solve the following linear equations using matrices, then check your answer:

A $4x + 3y = 26$, $5x - y = 4$

B $2x - 7y = 3$, $x - 3y = 2$

C $2x = 3 + 7y$, $y = 5 - x$

D $2y = 5 - x$, $2x = 3 - y$

- ⑦ **Geometry:** The straight line whose equation $y + ax = c$ passes through the two points (1, 5) and (3, 1), use the matrix to find the constants a and c.



Unit 2

**Lesson (2-1)****Linear Inequalities****Lesson objectives**

Related Links



- ✚ Solve first degree inequality in a variable.
- ✚ Solve first degree inequality in two variables and determining the region of solution graphically.

Let a , b , and c are three real numbers:

- If $a \leq b$, then $a + c \leq b + c$ (c is +ve or -ve)
- If $a \leq b$, then $a \times c \leq b \times c$ (c is +ve)
- If $a \leq b$, then $a \times c \geq b \times c$ (c is -ve)

First: Solving inequalities of first degree in one variable**Example ①**

Show graphically the solution set of the inequality: $3x + 10 > 1$, then write it in the form of an interval where $x \in \mathbb{R}$

Solution

$$\therefore 3x + 10 > 1$$

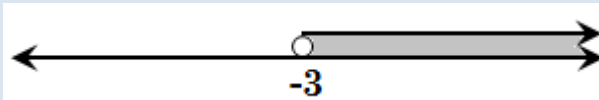
$$\therefore 3x > 1 - 10$$

$$\therefore 3x > -9$$

$$\therefore x > -\frac{9}{3}$$

$$\therefore x > -3$$

$$\therefore \text{The S.S.} =] -3, \infty[$$

**Example ②**

Show graphically the solution set of the inequality: $5x - 7 \leq 2x - 1$, then write it in the form of an interval where $x \in \mathbb{R}$

Solution

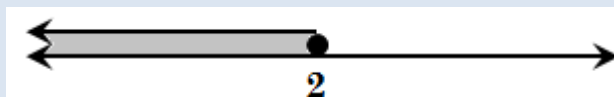
$$\therefore 5x - 7 \leq 2x - 1$$

$$\therefore 5x - 2x \leq -1 + 7$$

$$\therefore 3x \leq 6$$

$$\therefore x \leq 2$$

$$\therefore \text{The S.S.} =] -\infty, 2]$$



**Example 3**

Show graphically the S.S. of the inequality: $x - 1 \leq 4x + 5 < x + 17$, then write it in the form of an interval where $x \in \mathbb{R}$

Solution

$$\because x - 1 \leq 4x + 5 < x + 17$$

$$\therefore -1 \leq 4x + 5 < 17$$

$$\therefore -6 \leq 4x < 12$$

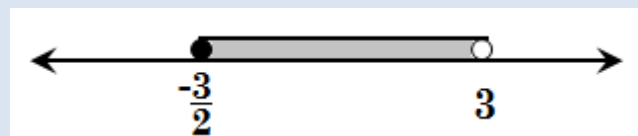
$$\therefore -\frac{3}{2} \leq x < 3$$

$$\therefore \text{The S.S.} = \left[-\frac{3}{2}, 3\right[$$

Add (- x)

$$\therefore -1 - 5 \leq 4x < 17 - 5$$

$$\therefore -\frac{6}{4} \leq x < \frac{12}{4}$$

**Example 4**

Find graphically the S.S. of the inequality: $2x - 2 \leq 3x - 1 < x + 5$ where $x \in \mathbb{R}$

Solution

$$\because 2x - 2 \leq 3x - 1$$

$$\therefore 2x - 3x \leq -1 + 2$$

$$\therefore -x \leq 1 \quad \times -1$$

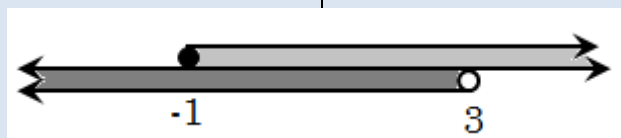
$$\therefore x \geq -1$$

$$\because 3x - 1 < x + 5$$

$$\therefore 3x - x < 5 + 1$$

$$\therefore 2x < 6$$

$$\therefore x < 3$$



$$\therefore \text{The S.S.} = [-1, 3[$$



Second: Solving inequalities of first degree in two variables graphically

For example: when represent the equation

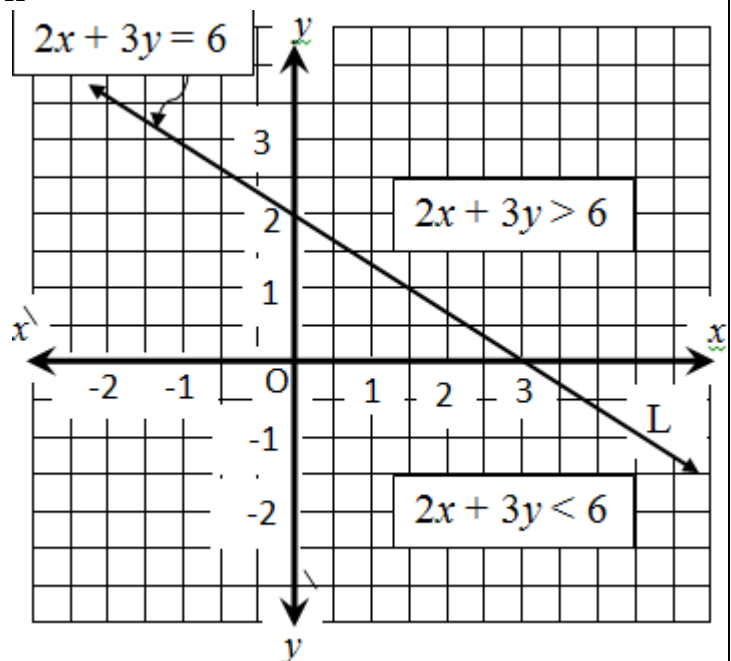
$2x + 3y = 6$ graphically, put $x = 0$

, then $y = 2$ and put $y = 0$, then $x = 3$

x	0	3
y	2	0

∴ The straight line L passing through the two points $(0, 2)$ and $(3, 0)$ is the graphic representation of the solution set of the equation

$$2x + 3y = 6$$



The straight line L divides the Cartesian plane into 3 sets of points:

① The set of points of the straight line L :

They are the set of all ordered pairs that satisfy its equation. This straight line is called the *boundary* line as the points $(0, 2)$, $(3, 0)$ and $(-1.5, 3)$

② The set of points of the Cartesian plane that lie on one side of straight line L :

and it is called the *half-plane* and it is denoted by S_1 as the points:

$(1, 3)$, $(2, 4)$, ... we notice that each point of these points makes the expression: $2x + 3y$ is greater than 6

③ The set of points of the Cartesian plane that lie on the other side of straight line L :

and it is called the *half-plane* also and it is denoted by S_2 as the points:

$(0, 0)$, $(1, 1)$, ... we notice that each point of these points makes the expression: $2x + 3y$ is smaller than 6



An important remark

① A *solid* (continuous) line if the inequality ($\leq$ or $\geq$)

② A *dashed* line if the inequality ($<$ or $>$)

Example ①

Represent graphically the S.S. of the inequality: $x - 3y \leq 3$ in $\mathbb{R} \times \mathbb{R}$

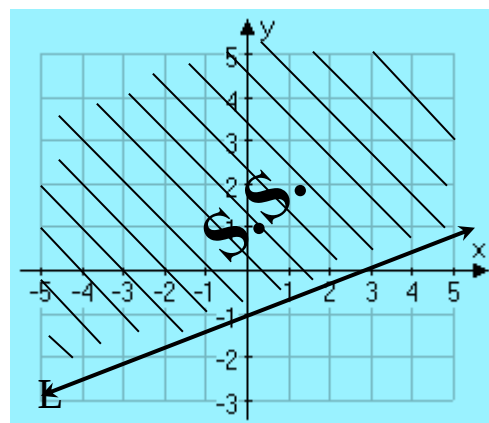
Solution

$$x - 3y \leq 3 \quad \text{Multiply by } (-1)$$

$$-x + 3y \geq -3$$

$$\text{L: } -x + 3y = -3$$

x	0	3
y	-1	0



S.S. = the shaded area

Example ②

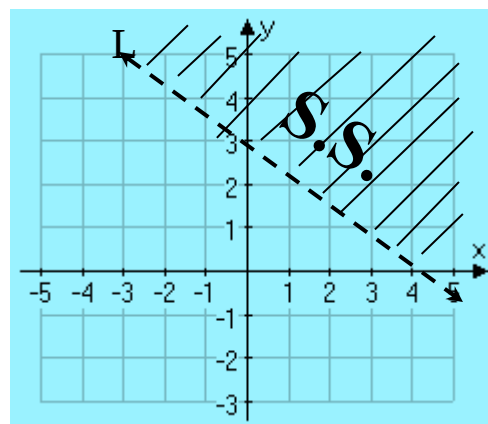
Represent graphically the S.S. of the inequality: $3x + 4y > 12$ in $\mathbb{R} \times \mathbb{R}$

Solution

$$3x + 4y > 12$$

$$\text{L: } 3x + 4y = 12$$

x	0	4
y	3	0



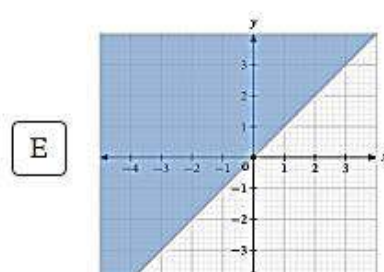
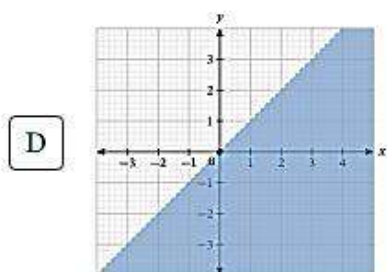
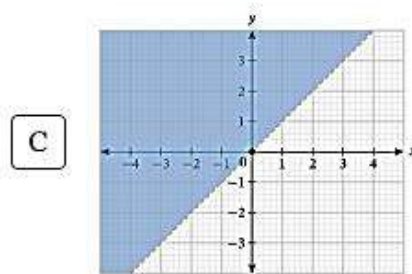
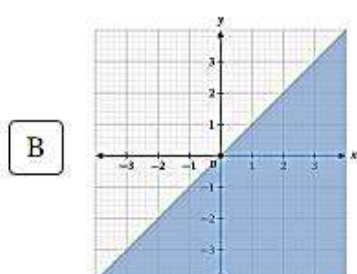
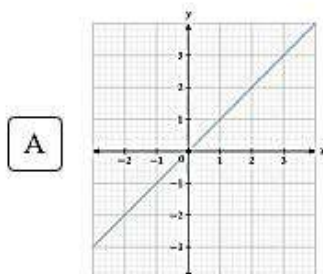
S.S. = the shaded area



PRACTICE

Question 1

Which of the following figures is the correct graph of $y = x$?



Question 2

Which inequality has been graphed in the given figure?

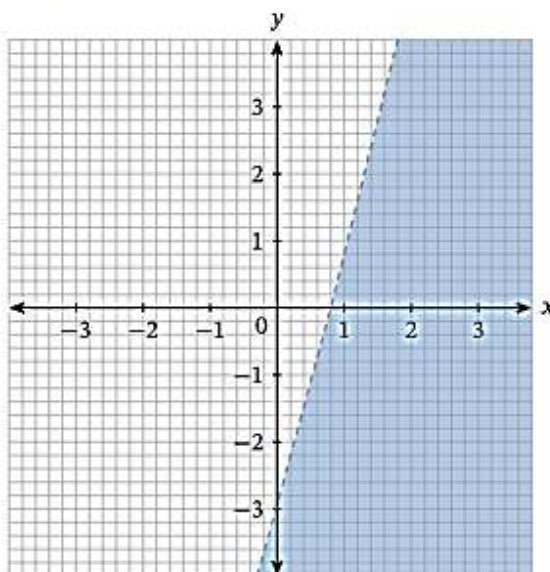
A $y \geq 4x - 3$

B $y > 4x - 3$

C $y < -3x + 4$

D $y \leq -3x + 4$

E $y < 4x - 3$

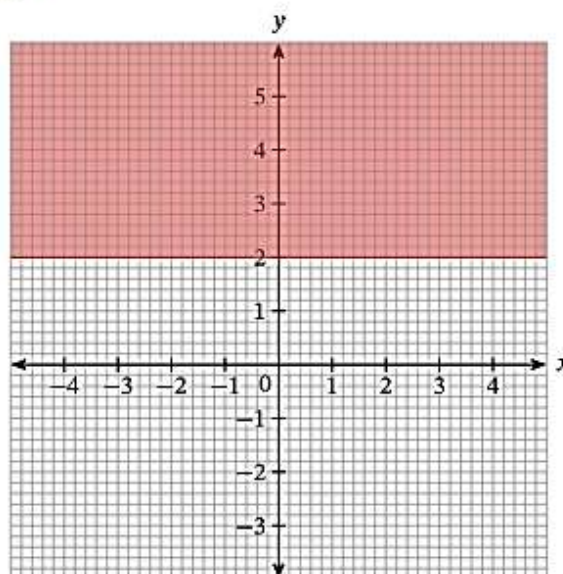




Question 3

What is the inequality graphed in the given figure?

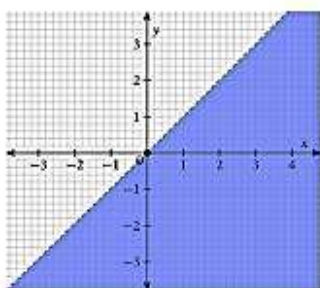
- ☐ A $y > 2$
- ☐ B $y < 2$
- ☐ C $y \leq 0$
- ☐ D $y \geq 2$
- ☐ E $y \leq 2$



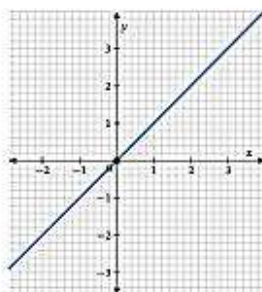
Question 4

Which of the following figures is the correct graph of $y > x$?

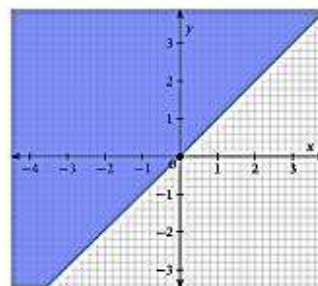
A



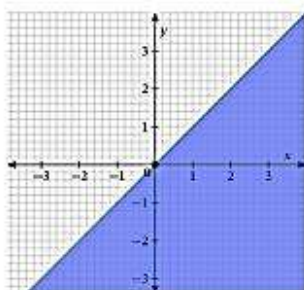
B



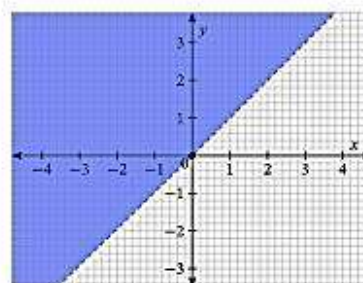
C



D



E

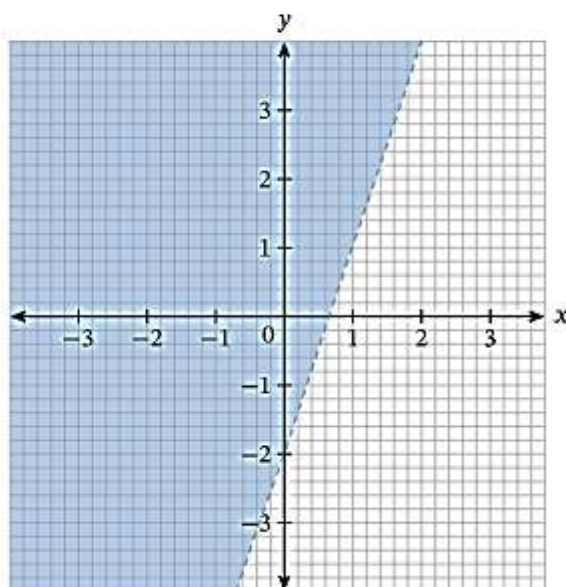




Question 5

Which inequality has been graphed in the given figure?

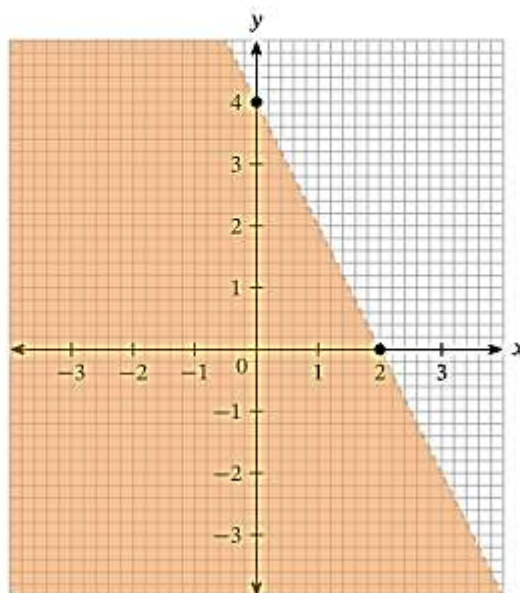
- ☐ A $y < -2x + 3$
- ☐ B $y \geq 3x - 2$
- ☐ C $y < 3x - 2$
- ☐ D $y \leq -2x + 3$
- ☐ E $y > 3x - 2$



Question 6

Which inequality has been graphed in the given figure?

- ☐ A $y < -2x + 4$
- ☐ B $y > -2x + 4$
- ☐ C $y < 4x - 2$
- ☐ D $y \leq -2x + 4$
- ☐ E $y \geq 4x - 2$

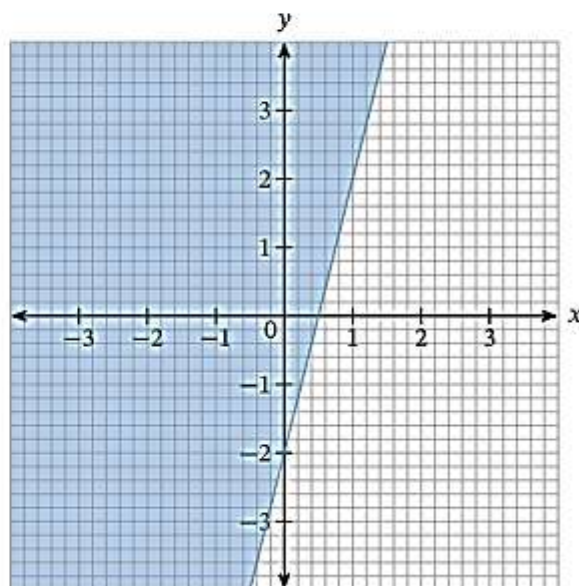




Question 7

Which inequality has been graphed in the given figure?

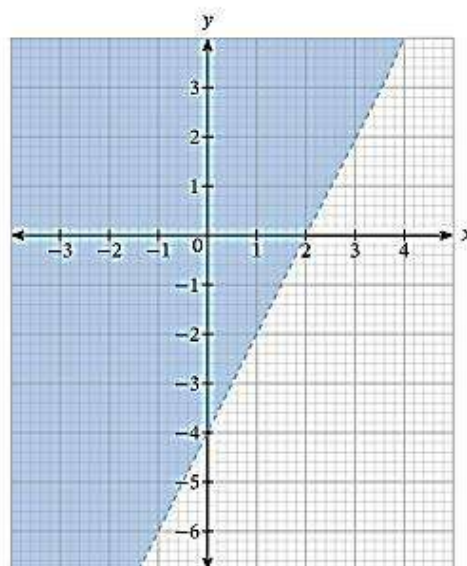
- ☐ A $y \geq -2x + 4$
- ☐ B $y \leq 4x - 2$
- ☐ C $y \geq 4x - 2$
- ☐ D $y > 4x - 2$
- ☐ E $y < -2x + 4$



Question 8

Which inequality has been graphed in the given figure?

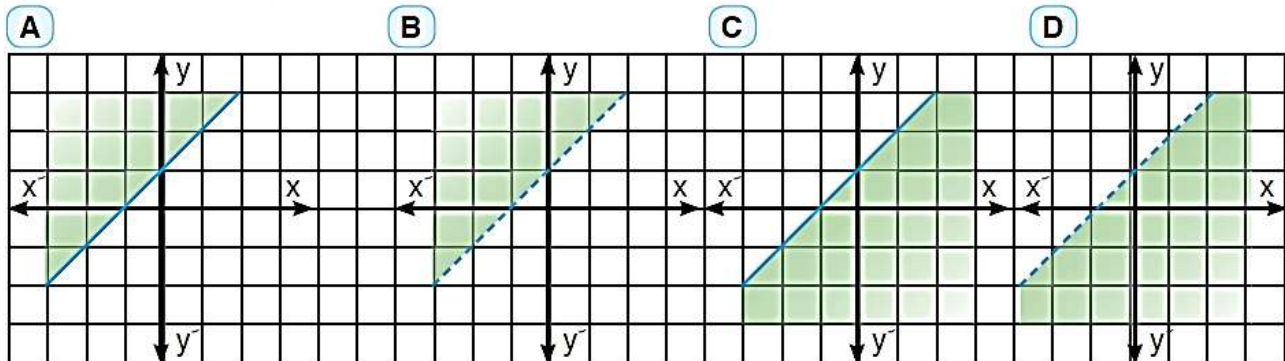
- ☐ A $y \geq 2x - 4$
- ☐ B $y \leq -4x + 2$
- ☐ C $y < 2x - 4$
- ☐ D $y > 2x - 4$
- ☐ E $y > -4x + 2$





Home Work

- 1 Join each inequality with the graph which represents its solution set (test the point $(0, 0)$ in each inequality).



1- $y \leq x + 1$

2- $y < x + 1$

3- $y > x + 1$

4- $y \geq x + 1$

- 2 Test which of the points is the solution of the inequality:

A $y \geq 2x + 3$ [$(0, 1)$, $(3, 9)$, $(-1, 0)$].

B $y < 2x + 3$ [$(0, 1)$, $(3, 9)$, $(-1, 0)$].

- 3 Find the solution set of each of the following inequalities:

A $y \leq x + 2$

B $y > 2x - 3$

C $x + 3y \leq 6$

- 4 **Consumer:** Suppose you want to buy decoration paper to decorate your class for a party to the excellent students. If the price of a roll of the golden decoration paper was 5 pounds and the price of a roll of the blue decoration paper was 3 pounds. If you pay at most 48 pounds to buy the decoration paper. How many rolls of each type can you buy? Explain your answer.

**Lesson (2-2)****Solving system of linear inequalities graphically****Lesson objectives****Related Links**

- Solve a system of linear inequality graphically.
- Solve life problems on systems of linear inequalities.

**Remarks**

- The equation: $y = 0$ is represented by x -axis
- The equation: $x = 0$ is represented by y -axis
- The equation: $y = a$ is represented by a straight line parallel to the x -axis and passes through the point $(0, a)$
- The equation: $x = a$ is represented by a straight line parallel to the y -axis and passes through the point $(a, 0)$

Example 1

** Represent graphically the S.S. of each of the following inequalities in a separate graph in $\mathbb{R} \times \mathbb{R}$

① $x > -2$

② $y \leq 1$

③ $x > y$

Solution

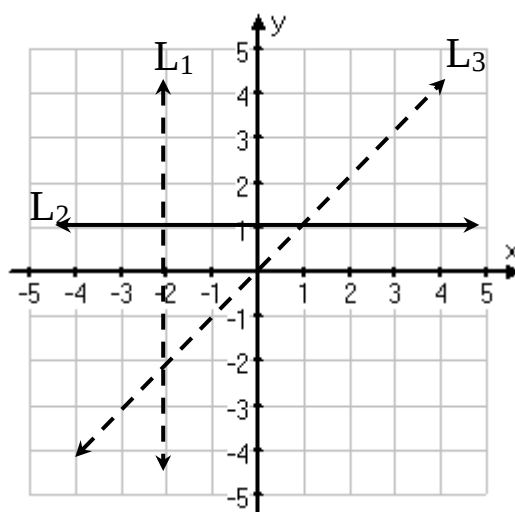
$L_1: x = -2$

$L_2: y = 1$

$x > y \Rightarrow y - x < 0$

$L_3: y - x = 0$

x	0	1
y	0	1



S.S. = Answer by yourself



Graphing solution of two or more inequalities of first degree in two variables

Example ②

Represent graphically the S.S. of each of the two inequalities:

$$x + 3y \leq 3, \quad 2x + y \leq 4 \quad \text{in } \mathbb{R} \times \mathbb{R}$$

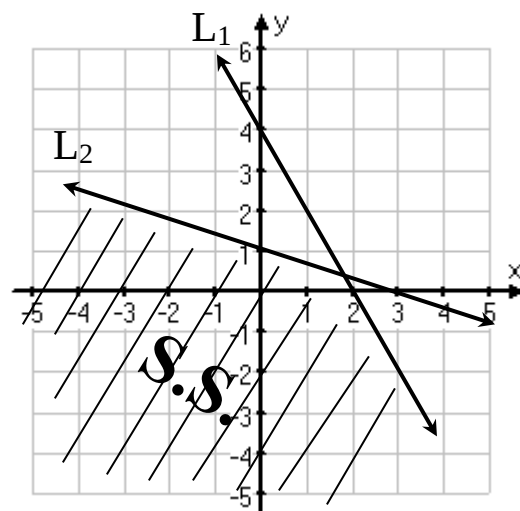
Solution

$$L_1: x + 3y = 3$$

x	0	3
y	1	0

$$L_2: 2x + y = 4$$

x	0	2
y	4	0



S.S. = the shaded area

Example ③

** Represent graphically the S.S. of the inequalities:

$$x \geq 0, \quad y \geq 0, \quad y + 3x \leq 9 \text{ and } y - x < 1 \quad \text{in } \mathbb{R} \times \mathbb{R}$$

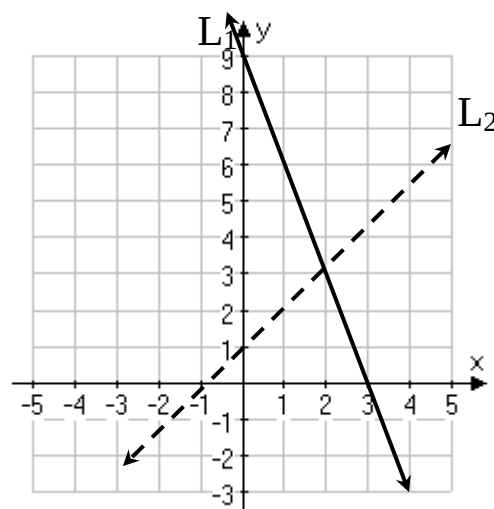
Solution

$$L_1: y + 3x = 9$$

x	0	3
y	9	0

$$L_2: y - x = 1$$

x	0	-1
y	1	0



S.S. = Answer by yourself

**Example 4**

** Represent graphically the S.S. of the inequalities:

$$-1 < x \leq 3, y \geq -1 \text{ and } 5y - 2x \leq 14 \text{ in } \mathbb{R} \times \mathbb{R}$$

Solution

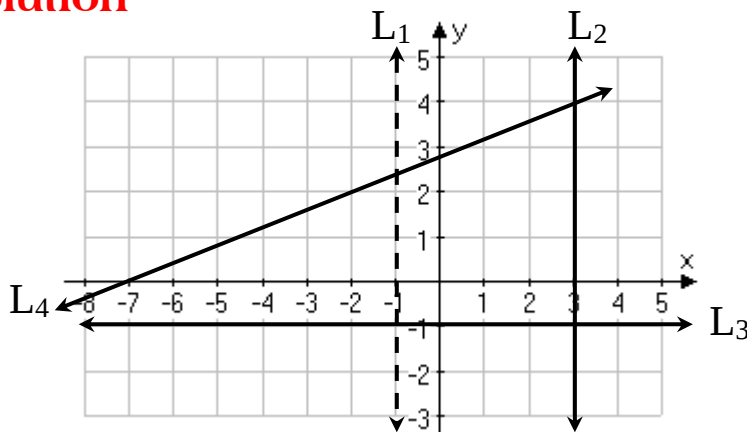
$$L_1: x = -1$$

$$L_2: x = 3$$

$$L_3: y = -1$$

$$L_4: 5y - 2x = 14$$

x	0	-7
y	$\frac{14}{5} = 2.8$	0



S.S. = Answer by yourself

Example 5

** Represent graphically the S.S. of the inequalities:

$$3x + 2y \leq 6, y + 3 \geq 0 \text{ and } x - y > 0 \text{ in } \mathbb{R} \times \mathbb{R}$$

Solution

$$L_1: 3x + 2y = 6$$

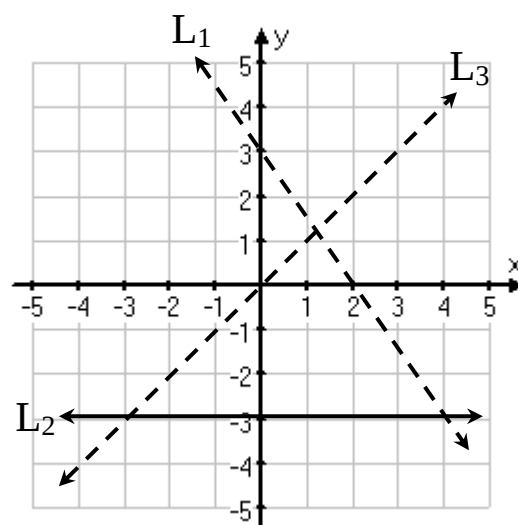
x	0	2
y	3	0

$$L_2: y = -3$$

$$x - y > 0 \Rightarrow y - x < 0$$

$$L_3: y - x = 0$$

x	0	1
y	0	1



S.S. = Answer by yourself

**Example 6**

Represent graphically the S.S. of the inequalities:

$$4y \geq 6x, \quad -3x + 2y < -6 \quad \text{in } \mathbb{R} \times \mathbb{R}$$

Solution

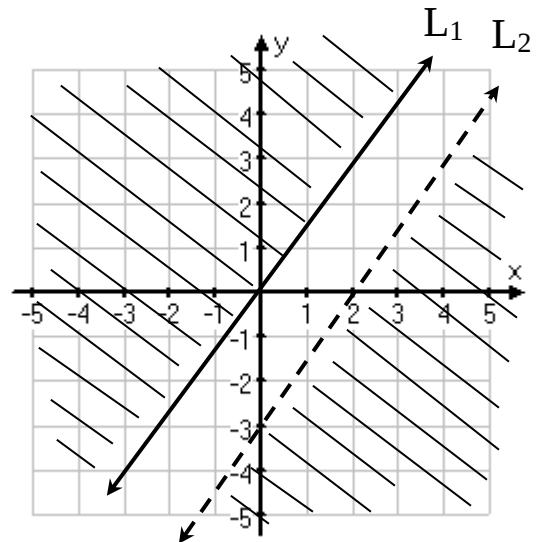
$$L_1: 4y = 6x$$

x	0	1
y	0	$\frac{3}{2} = 1.5$

$$L_2: -3x + 2y = -6$$

x	0	2
y	-3	0

$$\text{S.S.} = \emptyset$$



Home Work

① Solve each system of the following linear inequalities graphically:

A $x \geq 4$

$$y > x + 2$$

$$x + 2y \geq -2$$

B $y - x > 0$

$$2x + 2y \geq 12$$

$$y < 6 + 2x$$

C $x + 4y > 4$

$$4x + y \geq 2$$

$$x - y < 1$$

② Solve the following system graphically: $3x + 5y \geq 15, y < x - 1$

③ Find the solution of the following system of linear inequalities graphically:

$$y \leq x \text{ \& } y \geq x + 1$$

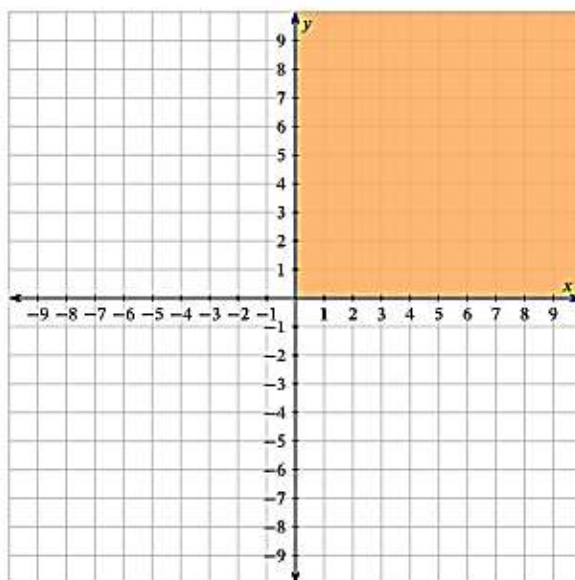


PRACTICE

Question 1

State the system of inequalities whose solution is represented by the following graph.

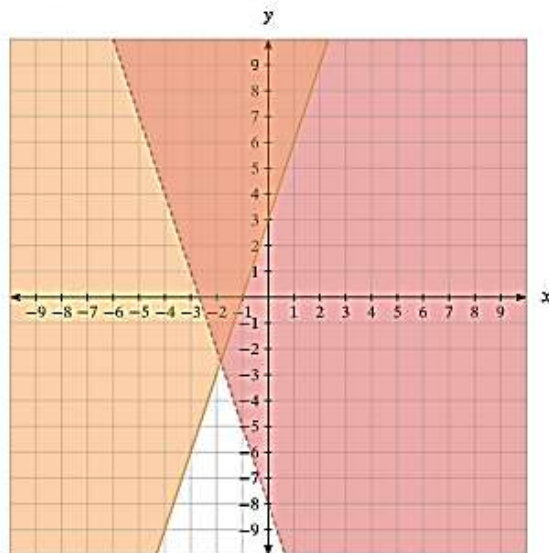
- ☐ A $x \leq 0, y \geq 0$
- ☐ B $x \geq 0, y \leq 0$
- ☐ C $x \leq 0, y \leq 0$
- ☐ D $x \geq 0, y \geq 0$



Question 2

State the system of inequalities whose solution is represented by the following graph.

- ☐ A $y > -3x + 3, y \geq 3x - 8$
- ☐ B $y \geq 3x - 3, y > -3x + 8$
- ☐ C $y \geq 3x + 3, y < -3x - 8$
- ☐ D $y < 3x - 3, y \leq -3x + 8$
- ☐ E $y \geq 3x + 3, y > -3x - 8$

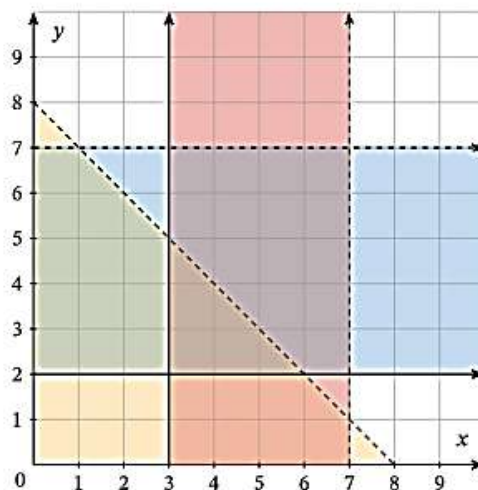




Question 3

Which of the points below belongs to the solution set of the system of inequalities represented by the figure shown?

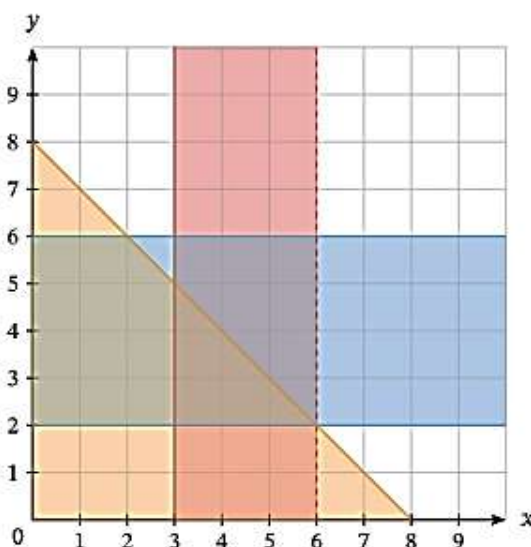
- A (6, 8)
- B (1, 3)
- C (3, 6)
- D (4, 3)



Question 4

State the system of inequalities whose solution is represented by the following graph.

- A $x \geq 0, y \geq 0, 3 \leq x \leq 6, 2 \leq y < 6, x + y < 8$
- B $x \geq 0, y \geq 0, 3 \leq x < 6, 2 \leq y \leq 6, x + y > 8$
- C $x \geq 0, y \geq 0, 3 \leq y < 6, 2 \leq x \leq 6, x + y \leq 8$
- D $x \geq 0, y \geq 0, 3 \leq x < 6, 2 \leq y \leq 6, x + y \leq 8$
- E $x \geq 0, y \geq 0, 3 \leq x \leq 6, 2 \leq y \leq 6, x + y \leq 8$

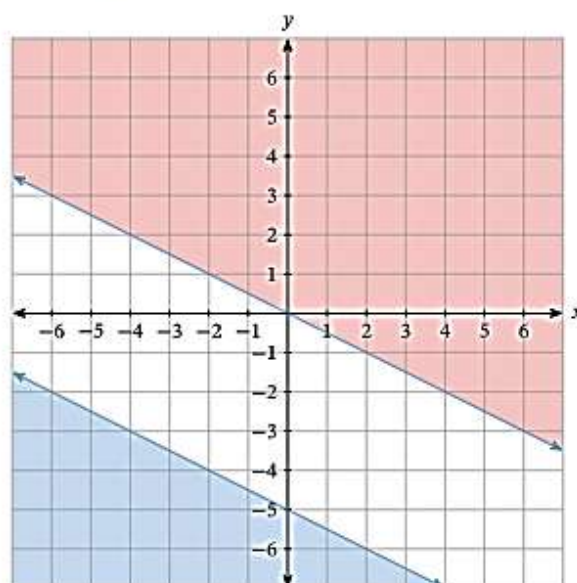




Question 5

Find the solution set of the linear inequalities shown below.

- ☐ A $y < -0.5x - 5$
☐ B $\emptyset$
☐ C $\mathbb{R}$
☐ D $y > -0.5x$



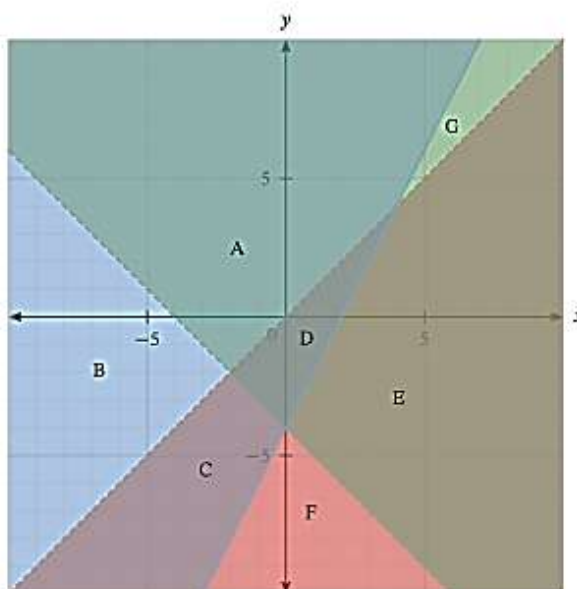
Question 6

Which regions on the graph contain solutions that satisfy both of the inequalities

$$y < x,$$

$$y \geq 2x - 4?$$

- ☐ A A and B
☐ B D and E
☐ C C and F
☐ D C and D
☐ E E and F



**Lesson (2-3)****Linear programming and optimization****Lesson objectives****Related Links**

- Find the max. & min. value to a function in a certain region.
- Use the linear programming to solve some problems.
- Record the data of mathematical life problem in a suitable table.

Example 1

Determine the S.S. of the following inequalities simultaneously:

$$x \geq 0, y \geq 0, x + 2y \leq 8 \text{ and } 3x + 2y \leq 12$$

Then find from the S.S. (x, y) that makes "P" maximum where $P = 50x + 75y$

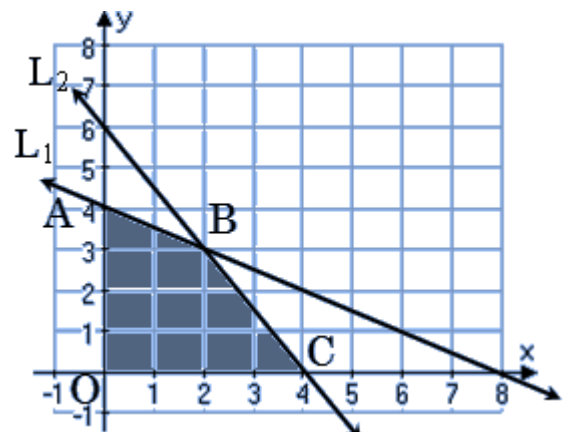
Solution

$$L_1: x + 2y = 8$$

x	0	8
y	4	0

$$L_2: 3x + 2y = 12$$

x	0	4
y	6	0



The optimum solution lies in the shaded area

$$A = (0, 4), \quad B = (2, 3)$$

$$C = (4, 0), \quad O = (0, 0)$$

$$\therefore P = 50x + 75y$$

$$\therefore P_A = 50(0) + 75(4) = 300$$

$$\& P_B = 50(2) + 75(3) = 325$$

$$\& P_C = 50(4) + 75(0) = 200$$

$$\& P_O = 50(0) + 75(0) = 0$$

$\therefore$ The point $B = (2, 3)$ makes "P" maximum.

**Example 2**

Determine the S.S. of the following inequalities simultaneously:

$$x \geq 0, y \geq 0, y - x \leq 3 \text{ and } 2y + 5x \leq 20$$

Then find from the S.S. the value of (x, y) that makes **(P)** maximum value where $P = 5x + 3y$

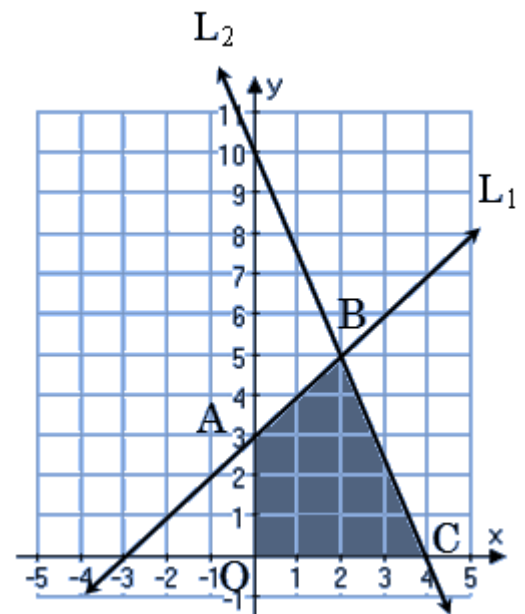
Solution

$$L_1: y - x = 3$$

x	0	-3
y	3	0

$$L_2: 2y + 5x = 20$$

x	0	4
y	10	0



The optimum solution lies in the shaded area

$$A = (0, 3), \quad B = (2, 5)$$

$$C = (4, 0), \quad O = (0, 0)$$

$$\therefore P = 5x + 3y$$

$$\therefore P_A = 5(0) + 3(3) = 9$$

$$\& P_B = 5(2) + 3(5) = 25$$

$$\& P_C = 5(4) + 3(0) = 20$$

$$\& P_O = 5(0) + 3(0) = 0$$

$\therefore$ the point $B = (2, 5)$ makes "P" maximum.

**Example ③**

If: $x \geq 0, y \geq 0, x + y \leq 4$ and $x + y \geq 3$:

Find the space of solution of the previous inequalities if $P = 15x + 10y$

Find the smallest value of the number P

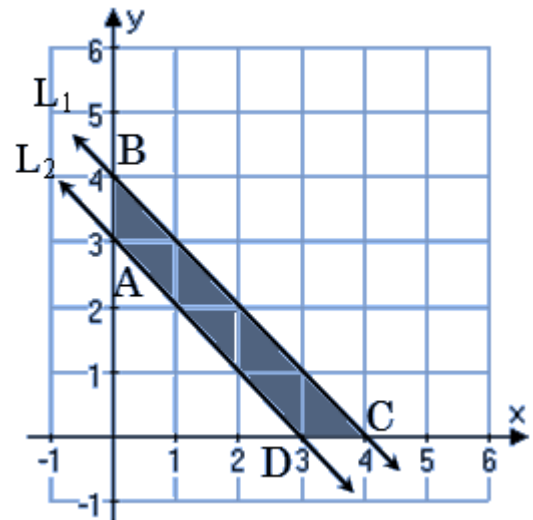
Solution

$L_1: x + y = 4$

x	0	4
y	4	0

$L_2: x + y = 3$

x	0	3
y	3	0



The optimum solution lies in the shaded area

$A = (0, 3)$, $B = (0, 4)$

$C = (4, 0)$, $D = (3, 0)$

$\therefore P = 15x + 10y$

$\therefore P_A = 15(0) + 10(3) = 30$

$\& P_B = 15(0) + 10(4) = 40$

$\& P_C = 15(4) + 10(0) = 60$

$\& P_D = 15(3) + 10(0) = 45$

$\therefore$ the point $A = (0, 3)$ makes "P" minimum.

**Example 4**

baby food factory produces two types of food with specifications, if the first type contains 2 units of vitamin (A), 3 units of vitamin (B), and the second type contains 3 units of vitamin (A), 2 units of vitamin B. If the child in its own food needs at least 120 units of vitamin (A), 100 units of vitamin (B), and the cost of the type (A) is 5 pounds, and the cost of type (B) is 4 pounds. What quantity to be purchased from each of the two types to achieve what the child needs in own food at the lowest possible cost?

Solution

Let: x be the number of goods of the 1st type
and y be the number of goods of the 2nd type

$$x \geq 0$$

$$y \geq 0$$

	Vitamin (A)	Vitamin (B)
x	2	3
y	3	2
	120	100

$$2x + 3y \geq 120 \quad \text{and} \quad 3x + 2y \geq 100$$

$$P = 5x + 4y \quad (\text{minimum})$$

$$L_1: 2x + 3y = 120$$

x	0	60
y	40	0

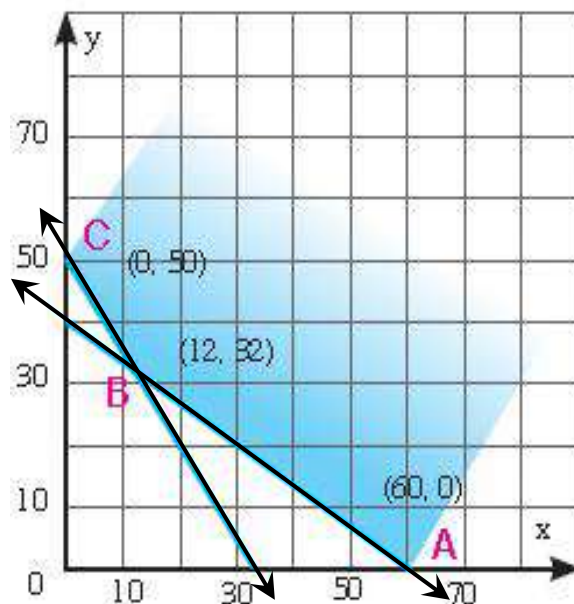
$$L_2: 3x + 2y = 100$$

x	0	33.3
y	50	0

$$A = (60, 0)$$

$$B = (12, 32)$$

$$C = (0, 50)$$



$$\therefore P = 5x + 4y$$

$$\therefore P_A = 5(60) + 4(0) = 300$$

$$\& P_B = 5(12) + 4(32) = 188$$

$$\& P_C = 5(0) + 4(50) = 200$$

$\therefore$ The point $B = (12, 32)$ makes "P" minimum.

Then the cost is to be minimum at B, the number of goods of the first type is 12 and the number of goods of the second type is 32

**Example 5**

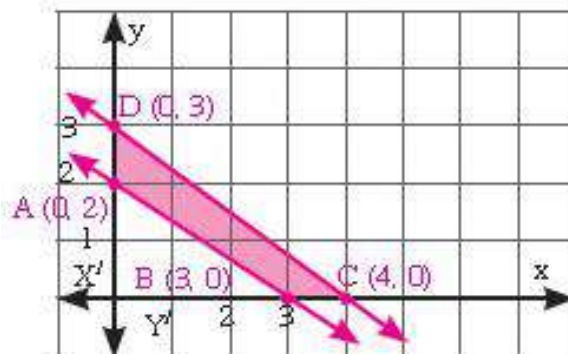
A factory produces two types of iron sheet offices, one of the workers collect each type and then another worker paint it. The first worker takes 2 hours to collect a unit of the first type, and 3 hours to collect a unit of the second type. While the second worker takes $1\frac{1}{2}$ hours to paint a unit of the first type, and 3 hours to paint a unit of the second type, if the first worker work at least 6 hours daily, while the second worker works at most 6 hours daily, the profit of the factory is 50 pounds to each unit of the two types. What is the number of units should the factory produce daily from each of the two types to achieve the maximum possible profit?

Solution

Let: x be the number of units of the 1st type
and y be the number of units of the 2nd type

$$x \geq 0$$

	Collect	Paint
x	2	$1\frac{1}{2}$
y	3	3
	6	6



$$2x + 3y \geq 6 \quad \text{and} \quad 1\frac{1}{2}x + 3y \leq 6$$

$$P = 50x + 50y \quad (\text{maximum})$$

$$L_1: 2x + 3y = 6$$

x	0	2
y	3	0

$$A = (0, 2)$$

$$B = (3, 0)$$

$$L_2: 1\frac{1}{2}x + 3y = 6$$

x	0	4
y	2	0

$$C = (4, 0)$$

$$D = (0, 3)$$

$$\therefore P = 50x + 50y$$

$$\therefore P_A = 50(0) + 50(2) = 100$$

$$\& P_B = 50(3) + 50(0) = 150$$

$$\& P_C = 50(4) + 50(0) = 200$$

$$\& P_D = 50(0) + 50(3) = 150$$

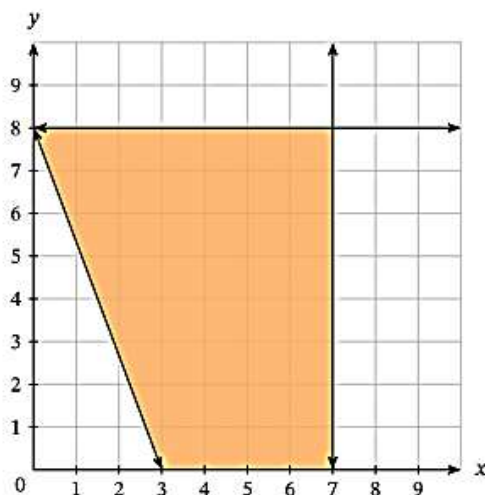
$\therefore$ The point $C = (4, 0)$ makes "P" maximum.

Then the profit is to be maximum at C, the number of units of first type is 4 units only.

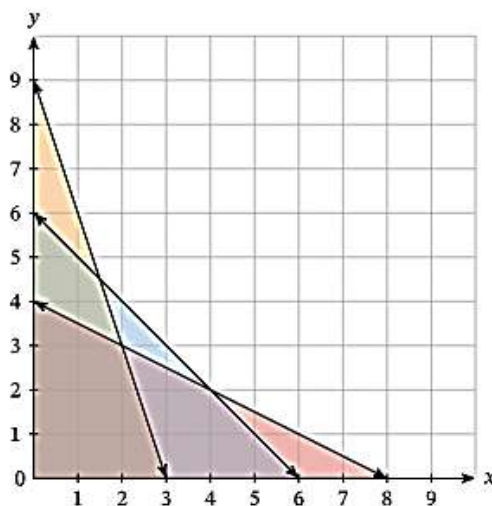
**PRACTICE****Question 1**

Determine the values of x and y that maximize the function $p = 5x + 2y$. Write your answer as a point (x, y) .

- ☐ A (0, 8)
- ☐ B (7, 8)
- ☐ C (7, 0)
- ☐ D (3, 0)

**Question 2**

Find the maximum value of the objective function $p = 2x + 6y$ given the constraints $x \geq 0$, $y \geq 0$, $x + y \leq 6$, $3x + y \leq 9$, and $x + 2y \leq 8$.

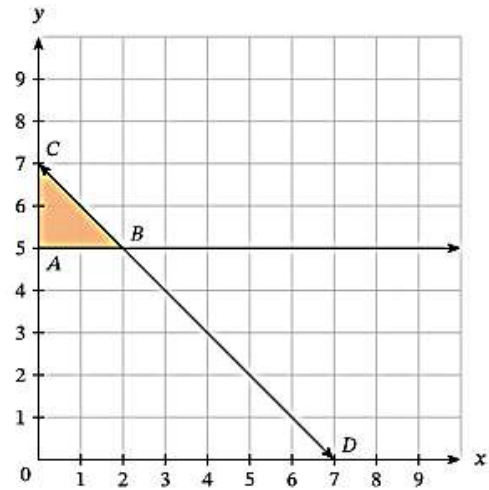




Question 3

Given the graph below and that $x \geq 0$, $y \geq 0$, $x + y \leq 7$, and $y \geq 5$, determine at which point the function $p = 3x - y$ has its maximum using linear programming.

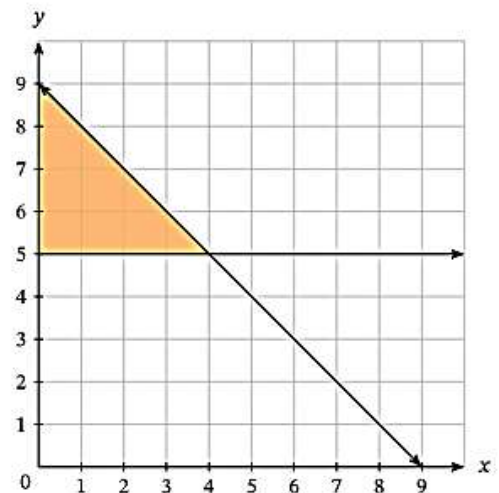
- ☐ A A
- ☐ B D
- ☐ C C
- ☐ D B



Question 4

Using linear programming, find the minimum and maximum values of the function $p = 4x - 3y$ given that $x \geq 0$, $y \geq 0$, $x + y \leq 9$, and $y \geq 5$.

- ☐ A The minimum value is -27 , and the maximum value is 1 .
- ☐ B The minimum value is -15 , and the maximum value is 1 .
- ☐ C The minimum value is -27 , and the maximum value is -15 .
- ☐ D The minimum value is 0 , and the maximum value is 9 .





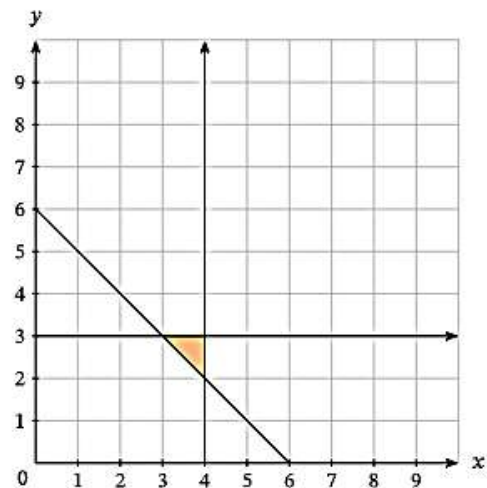
Question 5

A seafood restaurant sells two types of cooked fish; cod and eel. The restaurant sells NO LESS than 40 fish every day but it does not use more than 30 cod and no more than 45 eels. The price of one cod is 6 LE and that of an eel is 8 LE. Let x represent the amount of cod purchased each day, and y represent the amount of eel. Given that the manager wants to minimise the total price, p , of fish, state the objective function and the inequalities that will help the restaurant manager decide how many of each fish to buy.

- ☐ A $x \geq 0, y \geq 0, x + y \geq 40, x < 30, y < 45, p = 6x + 8y$
- ☐ B $x \geq 0, y \geq 0, x + y > 40, x \leq 30, y \leq 45, p = 6x + 8y$
- ☐ C $x \geq 0, y \geq 0, x + y \geq 40, x \geq 30, y \geq 45, p = 6x + 8y$
- ☐ D $x \geq 0, y \geq 0, x + y \geq 40, x \leq 30, y \leq 45, p > 6x + 8y$
- ☐ E $x \geq 0, y \geq 0, x + y \geq 40, x \leq 30, y \leq 45, p = 6x + 8y$

Question 6

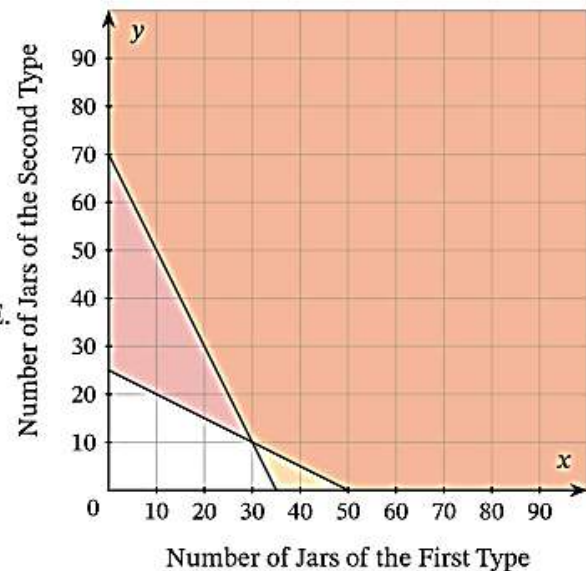
A candy store sells bags of marshmallows for 5 LE each and bags of cola candy for 6 LE each. A child wants to buy both types of candy and has restrictions on how many they can buy that are described by the figure shown, where x represents the number of bags of marshmallows they buy and y represents the number of bags of cola candy. What is the lowest price possible in this situation?





Question 7

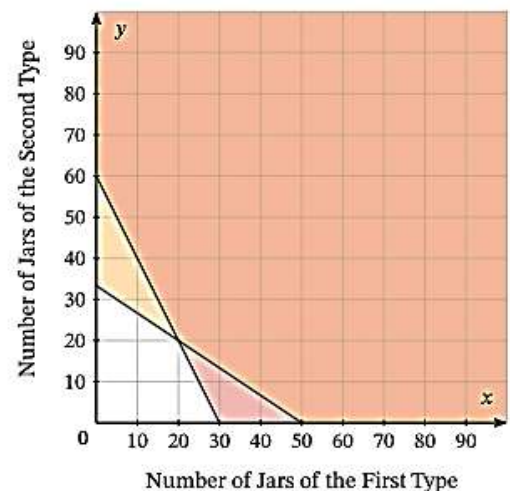
A baby food factory produces two types of baby food with different nutritional values. One jar of the first type has 2 units of vitamin A and 4 units of vitamin B, while a jar of the second type has 4 units of vitamin A and 2 units of vitamin B. Every child requires at least 100 units of vitamin A and 140 units of vitamin B each month. The first type costs 6 LE per jar while the second costs 4 LE per jar. Using the graph below, determine the objective function and then find the lowest possible cost required to supply a child with their required monthly nutrients.



- A $p = 4x + 6y$, and the lowest possible cost is 420 LE.
- B $p = 6x + 4y$, and the lowest possible cost is 280 LE.
- C $p = 2x + 4y$, and the lowest possible cost is 100 LE.
- D $p = 4x + 6y$, and the lowest possible cost is 1 800 LE.
- E $p = 6x + 4y$, and the lowest possible cost is 220 LE.

Question 8

A baby food factory produces two types of baby food with different nutritional values. One jar of the first type has 4 units of vitamin A and 2 units of vitamin B, while a jar of the second type has 2 units of vitamin A and 3 units of vitamin B. Every child requires at least 120 units of vitamin A and 100 units of vitamin B each month. The first type costs 6 LE per jar while the second costs 4 LE per jar. Using the graph below, determine how many of each type of jar should be bought to meet the child's monthly needs at the lowest possible cost.



- A Jars of the first type = 30, jars of the second type = 0
- B Jars of the first type = 20, jars of the second type = 20
- C Jars of the first type = 0, jars of the second type = 33
- D Jars of the first type = 0, jars of the second type = 60



Home Work

① Choose the correct answer from the given answers:

(A) The point which belongs to the solution of the enqualities: $x > 2$, $y > 1$, $x + y \geq 3$ is:

[(3, 1) , (1, 2) , (3, 2) , (1, 3)]

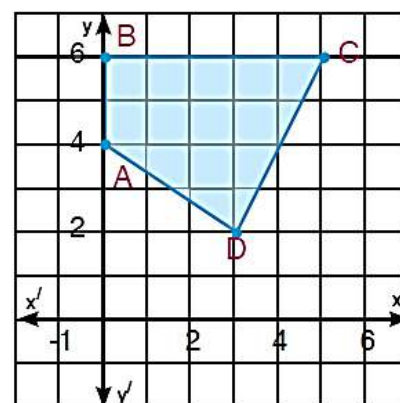
(B) The point at which the function $P = 40x + 20y$ has a maximum value is :

[(0, 0) , (0, -4) , (15, 10) , (25, 0)]

(C) The point at which the function $P = 35x + 10y$ has a minimum value is:

[(0, 0) , (0, 10) , (0, 40) , (20, 10)]

② Use the graph opposite to find the values of x, y which make the value of the objective function $p = 3x + 2y$ has a minimum value , then find this value .



③ Represent each of the following systems graphically, then find the maximum or minimum value to the objective function according to the given.

(A) $x + y \leq 5$

$y \geq 1$

$x \geq 2$

has minimum value to the objective
function $P = 2x + 3y$

(B) $2x + y \leq 6$

$x \geq 1$

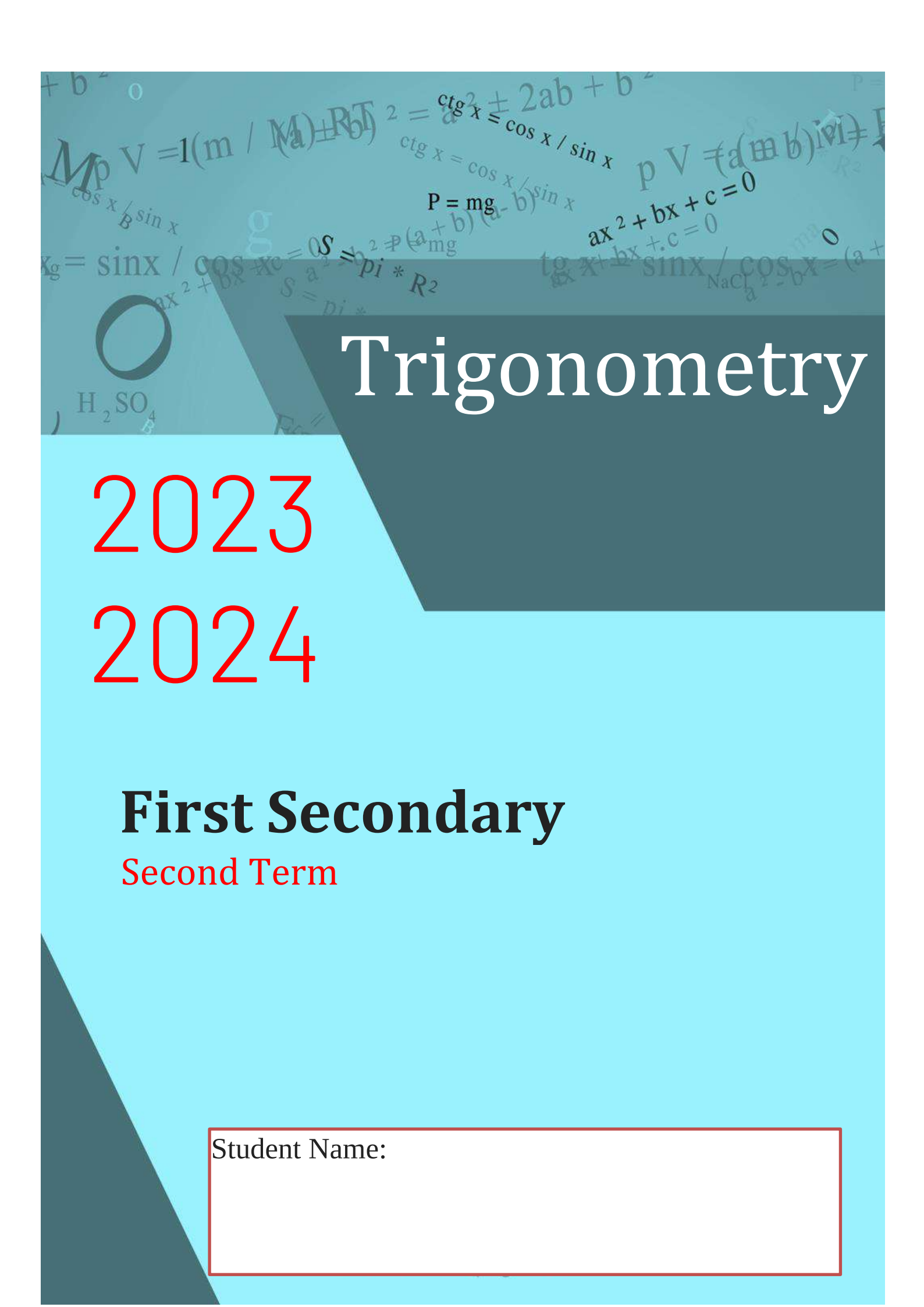
$y \geq 2$

has maximum value to the objective
function $P = 2x + 3y$

④ **Industry:** Suppose you manufacture and sell skin moisturizer, if manufacturing a unit of the normal moisturizer requires 2cm^3 of oil, 1cm^3 of cocoa butter, and manufacturing a unit of the excellent moisturizer requires 1cm^3 of oil, 2cm^3 of coca butter. You will gain 10 pounds for every unit of the normal kind, 8 pounds for every unit of the excellent kind. If you had 24 cm^3 of oil, 18cm^3 of cocoa butter. What is the number of units you can manufacture from each kind to get a maximum possible profit and what is this profit?



- ⑤ **Occupations:** A tailor has 10 metres of linen cloth and 6 metres of cotton cloth, he wants to make two types of clothing from his own available materials. The first kind of clothing, the tailor needs 1 metre of linen, 1 metre of cotton to gain 3 pounds profit, while the second kind of clothing he needs 2 metres of linen and 1 metre of cotton to gain 4 pounds profit. Find the number of clothing that should be made from each type such that the tailor gets the maximum profit
- ⑥ **Music:** One of the factories of musical instruments produces two types of blowing instruments, the first kind needs 25 units of copper, 4 units of nickel and the second type needs 15 units of copper, 8 units of nickel. If the available quantities in the factory on a day were 95 units of copper, 32 units of nickel and the profit of the factory from the first types was 60 pounds and 48 pounds from the second type. Find the number of instruments which the factory should produce from each type to get the maximum profit.
.....
- ⑦ **Tourism:** One of the tourism companies set up an air bridge to transport 1600 tourists and 90 tons of luggage with the least cost. If there are two types of aeroplanes A and B and the number of available aeroplanes from the first type was 12 aeroplanes, and 9 aeroplanes from the second type and the full load to the aeroplane of the first type A was 200 persons, 6 tons of luggage and the full load to the aeroplane of the second type B was 100 persons, 15 tons of luggage. The rent of the aeroplane of the type A was 320000 pounds, and the rent of the type B was 150000 pounds. How many aeroplanes from each type could the company rent?



Trigonometry

2023

2024

First Secondary

Second Term

Student Name:



**Lesson (5-1)****Trigonometric Identities****Lesson objectives****Related Links**

- Identify Concept of trigonometric identity.
- Simplify the trigonometric expressions.
- Prove the validity of trigonometric identity.

**The unit circle:**

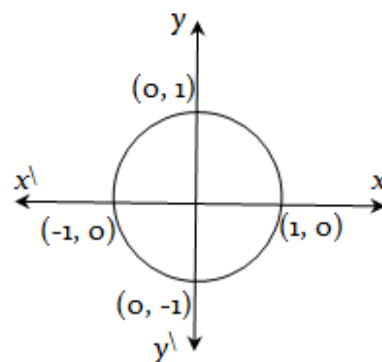
$$x^2 + y^2 = 1$$

$$\cos \theta = x$$

$$\sin \theta = y$$

$$\sin \theta = \frac{1}{\csc \theta} \quad \cos \theta = \frac{1}{\sec \theta} \quad \tan \theta = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$



$$\textcircled{1} \quad \sin^2 \theta + \cos^2 \theta = 1 \quad \sin^2 \theta = 1 - \cos^2 \theta \quad \cos^2 \theta = 1 - \sin^2 \theta$$

$$\div \sin^2 \theta$$

$$\textcircled{2} \quad 1 + \cot^2 \theta = \csc^2 \theta \quad \cot^2 \theta = \csc^2 \theta - 1 \quad \csc^2 \theta - \cot^2 \theta = 1$$

$$\div \cos^2 \theta$$

$$\textcircled{3} \quad 1 + \tan^2 \theta = \sec^2 \theta \quad \tan^2 \theta = \sec^2 \theta - 1 \quad \sec^2 \theta - \tan^2 \theta = 1$$

Remember that:

$$\textcircled{1} \quad \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\textcircled{2} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta \\ \csc(-\theta) &= -\csc \theta & \sec(-\theta) &= \sec \theta \\ \tan(-\theta) &= -\tan \theta & \cot(-\theta) &= -\cot \theta \end{aligned}$$

Before the beginning looks carefully page 15

**Example ①**

Prove that: $\tan x + \cot x = \sec x \csc x$

Proof

$$\begin{aligned} \text{L.H.S.} &= \tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \\ &= \frac{1}{\cos x \sin x} = \sec x \csc x = \text{R.H.S.} \end{aligned}$$

Example ②

Prove that: $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

Proof

$$\begin{aligned} \text{L.H.S.} &= (\sin x + \cos x)^2 + (\sin x - \cos x)^2 \\ &= \sin^2 x + 2 \sin x \cos x + \cos^2 x + \sin^2 x - 2 \sin x \cos x + \cos^2 x \\ &= \sin^2 x + \cos^2 x + \sin^2 x + \cos^2 x = 1 + 1 = 2 = \text{R.H.S.} \end{aligned}$$

Example ③

Prove that: $\cos^4 x - \sin^4 x = 1 - 2 \sin^2 x$

Proof

$$\begin{aligned} \text{L.H.S.} &= \cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) \\ &= 1 \times (\cos^2 x - \sin^2 x) \\ &= \cos^2 x - \sin^2 x \\ &= (1 - \sin^2 x) - \sin^2 x = 1 - 2 \sin^2 x = \text{R.H.S.} \end{aligned}$$

**Example ④**

Prove that: $\tan^2 x = \sin^2 x + \sin^2 x \tan^2 x$

Proof

$$\begin{aligned} \text{R.H.S.} &= \sin^2 x + \sin^2 x \tan^2 x = \sin^2 x (1 + \tan^2 x) \\ &= \sin^2 x \sec^2 x = \sin^2 x \times \frac{1}{\cos^2 x} \\ &= \frac{\sin^2 x}{\cos^2 x} = \tan^2 x = \text{L.H.S.} \end{aligned}$$

Example ⑤

Prove that: $\frac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$

Proof

$$\text{L.H.S.} = \frac{1 + \tan^2 x}{1 + \cot^2 x} = \frac{\sec^2 x}{\csc^2 x} = \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x = \text{R.H.S.}$$

Example ⑥

Prove that: $\frac{\cos^4 x - \cos^2 x}{\sin^4 x - \sin^2 x} = \sec^2 x - \tan^2 x$

Proof

$$\text{L.H.S.} = \frac{\cos^4 x - \cos^2 x}{\sin^4 x - \sin^2 x} = \frac{\cos^2 x (\cos^2 x - 1)}{\sin^2 x (\sin^2 x - 1)} = \frac{\cos^2 x (-\sin^2 x)}{\sin^2 x (-\cos^2 x)} = 1 \quad \Rightarrow \textcircled{1}$$

$$\text{R.H.S.} = \sec^2 x - \tan^2 x = 1 \quad \Rightarrow \textcircled{2}$$

From ① and ②: $\therefore \text{L.H.S.} = \text{R.H.S.}$

**Example 7**

Prove that: $\frac{2 \tan \theta}{1 + \tan^2 \theta} = 2 \sin \theta \cos \theta$

Proof

$$\begin{aligned} \text{L.H.S.} &= \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2 \tan \theta}{\sec^2 \theta} = 2 \tan \theta \cos^2 \theta = 2 \frac{\sin \theta}{\cancel{\cos \theta}} \times \cancel{\cos \theta}^2 \theta \\ &= 2 \sin \theta \cos \theta = \text{R.H.S.} \end{aligned}$$

Example 8

Prove that: $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = 2 \cos^2 \theta - 1$

Proof

$$\begin{aligned} \text{L.H.S.} &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{\sec^2 \theta} = \frac{\cos^2 \theta - \sin^2 \theta}{\cancel{\cos^2 \theta}} \times \cancel{\cos^2 \theta} = \cos^2 \theta - \sin^2 \theta \\ &= \cos^2 \theta - (1 - \cos^2 \theta) = \cos^2 \theta - 1 + \cos^2 \theta = 2 \cos^2 \theta - 1 = \text{R.H.S.} \end{aligned}$$

Example 9

Prove that: $\frac{\sin x \sin (90 - x)}{\tan x} + \frac{\tan(180 + x)}{\sec x \csc x} = 1$

Proof

$$\begin{aligned} \text{L.H.S.} &= \frac{\sin x \sin (90 - x)}{\tan x} + \frac{\tan(180 + x)}{\sec x \csc x} \\ &= \frac{\sin x \cos x}{\frac{\sin x}{\cos x}} + \frac{\tan x}{\sec x \csc x} = \frac{\cancel{\sin x} \cos x}{\cancel{\sin x}} + \frac{\sin x}{\cos x} \times \sin x \cancel{\cos x} \\ &= \cos^2 x + \sin^2 x = \text{R.H.S.} \end{aligned}$$



SCHOOL BOOK EXAMPLES

Notice that

$$\sin \theta \times \sin \theta = (\sin \theta)^2 = \sin^2 \theta$$

1 Write the simplest form: $(\sin \theta + \cos \theta)^2 - 2 \sin \theta \cos \theta$

Solution

A The expression $(\sin \theta + \cos \theta)^2 - 2 \sin \theta \cos \theta$
 $= \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta - 2 \sin \theta \cos \theta$
 $= \sin^2 \theta + \cos^2 \theta$
 $= 1$

remove the parentheses
Simplify
apply pythagorean identity:

1 Write in the simplest form: $\frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}$

Solution

the expression : $\frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}$
 $= \frac{\sec^2 \theta}{\csc^2 \theta}$ apply pythagorean identity
 $= \frac{1}{\cos^2 \theta} \div \frac{1}{\sin^2 \theta}$
 $= \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta$

2 Prove the validity of the identity: $\frac{\cos^2 \theta}{1 - \sin \theta} = 1 + \sin \theta$

Solution

L.H.S $= \frac{\cos^2 \theta}{1 - \sin \theta} = \frac{1 - \sin^2 \theta}{1 - \sin \theta}$
 $= \frac{(1 + \sin \theta)(1 - \sin \theta)}{1 - \sin \theta} = 1 + \sin \theta = \text{R.H.S}$

Remember

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \sin^2 \theta &= 1 - \cos^2 \theta \\ \cos^2 \theta &= 1 - \sin^2 \theta \end{aligned}$$



③ Prove the validity of the identity : $\tan \theta + \cot \theta = \sec \theta \csc \theta$

 **Solution**

$$\begin{aligned}\text{L.H.S} &= \tan \theta + \cot \theta \\ &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \\ &= \frac{1}{\cos \theta \sin \theta} \\ &= \sec \theta \csc \theta = \text{R.H.S}\end{aligned}$$

④ Prove the validity of the identity: $\frac{1 - \cot^2 \theta}{1 + \cot^2 \theta} = 2\sin^2 \theta - 1$

 **Solution**

$$\begin{aligned}\text{L.H.S} &= \frac{1 - \cot^2 \theta}{1 + \cot^2 \theta} \\ &= \frac{1 - \cot^2 \theta}{\csc^2 \theta} = \frac{1 - \frac{\cos^2 \theta}{\sin^2 \theta}}{\frac{1}{\sin^2 \theta}} \quad \text{change to sin } \theta, \cos \theta \\ &= \frac{1 - \frac{\cos^2 \theta}{\sin^2 \theta}}{\frac{1}{\sin^2 \theta}} \times \frac{\sin^2 \theta}{\sin^2 \theta} = \sin^2 \theta - \cos^2 \theta \\ &= \sin^2 \theta - (1 - \sin^2 \theta) \\ &= 2\sin^2 \theta - 1 = \text{R.H.S}\end{aligned}$$



SCHOOL BOOK EXERCISES SELECTED

① Prove that: $\frac{\sin \theta \cos \theta}{\tan \theta} + \frac{\tan \theta}{\sec \theta \csc \theta} = 1$

Proof

$$\begin{aligned} \text{L.H.S.} &= \frac{\sin \theta \cos \theta}{\tan \theta} + \frac{\tan \theta}{\sec \theta \csc \theta} = \frac{\sin \theta \cos \theta \times \sec \theta \csc \theta + \tan \theta \times \tan \theta}{\tan \theta \sec \theta \csc \theta} \\ &= \frac{1 + \tan^2 \theta}{\frac{\sin \theta}{\cos \theta} \times \frac{1}{\cos \theta} \times \frac{1}{\sin \theta}} = \frac{\sec^2 \theta}{\frac{1}{\cos^2 \theta}} = \frac{\sec^2 \theta}{\sec^2 \theta} = 1 = \text{R.H.S.} \end{aligned}$$

② Prove that: $\frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} + \frac{\sin^3 \theta - \cos^3 \theta}{\sin \theta - \cos \theta} = 2$

Proof

$$\begin{aligned} \text{L.H.S.} &= \frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} + \frac{\sin^3 \theta - \cos^3 \theta}{\sin \theta - \cos \theta} \\ &= \frac{(\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta)}{(\sin \theta + \cos \theta)} + \frac{(\sin \theta - \cos \theta)(\sin^2 \theta + \sin \theta \cos \theta + \cos^2 \theta)}{(\sin \theta - \cos \theta)} \\ &= 1 - \sin \theta \cos \theta + 1 + \sin \theta \cos \theta = 2 = \text{R.H.S.} \end{aligned}$$

③ Prove that: $\frac{1 + \cot^2 \theta}{1 + \tan^2 \theta} = \cot^2 \theta$

Proof

$$\text{L.H.S.} = \frac{1 + \cot^2 \theta}{1 + \tan^2 \theta} = \frac{\csc^2 \theta}{\sec^2 \theta} = \frac{\frac{1}{\sin^2 \theta}}{\frac{1}{\cos^2 \theta}} = \frac{\cos^2 \theta}{\sin^2 \theta} = \cot^2 \theta = \text{R.H.S.}$$

④ Prove that: $\frac{\tan \theta + \cot \theta}{\sec \theta \csc \theta} = 1$

Proof

$$\text{L.H.S.} = \frac{\tan \theta + \cot \theta}{\sec \theta \csc \theta} = \frac{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}}{\frac{1}{\sin \theta} \times \frac{1}{\cos \theta}} = \frac{\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \times \cos \theta}}{\frac{1}{\sin \theta \times \cos \theta}} = 1 = \text{R.H.S.}$$

**PRACTICE****Choose the correct answer**

Q (1): Simplify: $\frac{\sin^2 \theta + \cos^2 \theta}{\csc^2 \theta - \cot^2 \theta}$

- Ⓐ $-\cos^2 \theta$
- Ⓑ 1
- Ⓒ -1
- Ⓓ $\cos^2 \theta$

Q (2): Simplify: $\frac{1 - \sin^2 \theta}{\csc^2 \theta - \cot^2 \theta}$

- Ⓐ $-\sin^2 \theta$
- Ⓑ $\cos^2 \theta$
- Ⓒ $-\cos^2 \theta$
- Ⓓ $\sin^2 \theta$

Q (3): Simplify: $\frac{\sec^2 \theta - 1}{\sec^2 \theta - \tan^2 \theta}$

- Ⓐ $-\sin^2 \theta$
- Ⓑ $\tan^2 \theta$
- Ⓒ $-\tan^2 \theta$
- Ⓓ $\sin^2 \theta$

Q (4): Simplify: $(1 - \tan \theta)^2 + (1 + \tan \theta)^2$

- Ⓐ $2 \sec^2 \theta$
- Ⓑ $\csc^2 \theta$
- Ⓒ $\sec^2 \theta$
- Ⓓ $2 \csc^2 \theta$



Q (5): Simplify: $1 + \cot^2 (90^\circ - \theta)$

- Ⓐ $\tan^2 \theta$
- Ⓑ $\csc^2 \theta$
- Ⓒ $\sec^2 \theta$
- Ⓓ $\cot^2 \theta$

Q (6): Simplify: $(1 + \cot \theta)^2 - 2 \cot \theta$

- Ⓐ $\tan^2 \theta$
- Ⓑ $\csc^2 \theta$
- Ⓒ $\sec^2 \theta$
- Ⓓ $\cot^2 \theta$

Q (7): Simplify: $\frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}$

- Ⓐ -1
- Ⓑ $\cot^2 \theta$
- Ⓒ $\tan^2 \theta$
- Ⓓ 1

Q (8): Simplify: $\frac{1 + \cot^2 \left(\frac{3\pi}{2} - \theta\right)}{1 + \tan^2 \left(\frac{\pi}{2} - \theta\right)}$

- Ⓐ 1
- Ⓑ -1
- Ⓒ $\tan^2 \theta$
- Ⓓ $\cot^2 \theta$



Q (9): Simplify: $(1 + \cot \theta)^2 - 2 \cot \theta$

- Ⓐ $\tan^2 \theta$
- Ⓑ $\csc^2 \theta$
- Ⓒ $\sec^2 \theta$
- Ⓓ $\cot^2 \theta$

Q (10): Evaluate: $\tan^3 \theta + \cot^3 \theta$ given than: $\tan \theta + \cot \theta = 6$

- Ⓐ 210
- Ⓑ 198
- Ⓒ 222
- Ⓓ 216

Q (11): Simplify: $\frac{\sin \theta \cos \theta}{\tan \theta} + \frac{\tan \theta}{\sec \theta \csc \theta}$

- Ⓐ 1
- Ⓑ $\cot^2 \theta$
- Ⓒ -1
- Ⓓ $\tan^2 \theta$

Q (12): Simplify: $\cos^2 \theta \sec \theta \csc \theta$

- Ⓐ 1
- Ⓑ $\cot \theta$
- Ⓒ $\tan \theta$
- Ⓓ $\cos^2 \theta$



Q (13): Simplify: $\cos \theta \csc \theta \sin \theta$

- Ⓐ $\tan \theta$
- Ⓑ $\sin \theta$
- Ⓒ $\cos \theta$
- Ⓓ $-\cos^2 \theta$

Q (14): Simplify: $\frac{\tan \theta \sin \theta}{\sec \theta}$

- Ⓐ 1
- Ⓑ $-\cot^2 \theta$
- Ⓒ $-\sin^2 \theta$
- Ⓓ $\sin^2 \theta$

Q (15): Simplify: $\cos(180^\circ - \theta) \cos(90^\circ - \theta) \sec(\theta - 180^\circ)$

- Ⓐ 1
- Ⓑ $\sin \theta$
- Ⓒ $\cos \theta$
- Ⓓ $-\cos^2 \theta$

Q (16): Simplify: $\sin \theta \cos \theta (\tan \theta + \cot \theta)$

- Ⓐ $\tan \theta$
- Ⓑ 1
- Ⓒ $\cot \theta$
- Ⓓ -1



Q (17): Simplify: $\sin \theta \cos \theta \sin(90^\circ - \theta)$

- (a) $-\tan \theta$
- (b) $\tan \theta$
- (c) -1
- (d) 1

Q (18): Find the value: $\sec^2 \theta$ given that $\tan^2 \theta = 4$

- (a) $\frac{1}{4}$
- (b) $\frac{1}{5}$
- (c) 4
- (d) 5

Q (19): $\tan \theta \csc \theta = \dots\dots\dots$

- (a) 1
- (b) $\cos \theta$
- (c) $\sec \theta$
- (d) $\csc \theta$

Q (20): $\frac{\tan \theta \cot \theta}{\sec \theta} = \dots\dots\dots$

- (a) $\sin \theta$
- (b) $\cos \theta$
- (c) $\sec \theta$
- (d) $\csc \theta$

Q (21): $5 \cos^2 30^\circ + 5 \sin^2 30^\circ = \dots\dots\dots$

- (a) 5
- (b) 1
- (c) 25
- (d) 10



Q (22): $\sin^2(180^\circ - \theta) + \sin^2(270^\circ - \theta) = \dots\dots\dots$

- Ⓐ $2 \sin^2 \theta$
- Ⓑ $\cos^2 \theta$
- Ⓒ 1
- Ⓓ 10

Q (23): $(1 + \cot \theta)^2 - 2 \cot \theta = \dots\dots\dots$

- Ⓐ $\sec^2 \theta$
- Ⓑ $\cos^2 \theta$
- Ⓒ $\csc^2 \theta$
- Ⓓ $-\csc^2 \theta$

Q (24): $\frac{1 - \sin^2 \theta}{1 - \cos^2 \theta}$ in simplest form equals $\dots\dots\dots$

- Ⓐ 1
- Ⓑ -1
- Ⓒ $\tan^2 \theta$
- Ⓓ $\cot^2 \theta$

Q (25): $\frac{\sec^2 \theta - \tan^2 \theta}{\sin^2 \theta + \cos^2 \theta}$ in simplest form equals $\dots\dots\dots$

- Ⓐ 1
- Ⓑ -1
- Ⓒ $\tan^2 \theta$
- Ⓓ $\sec^2 \theta$

Q (26): $\sin^2 \theta + \cos^2 \theta + \tan^2 \theta = \dots\dots\dots$

- Ⓐ 1
- Ⓑ $\cot^2 \theta$
- Ⓒ $\sec^2 \theta$
- Ⓓ $\csc^2 \theta$



Home Work

First: Multiple choice

- ① $\frac{\tan\theta \cot\theta}{\sec\theta}$ equals: in the simplest form
 (A) $\sin\theta$ (B) $\cos\theta$ (C) $\sec\theta$ (D) $\csc\theta$
- ② $\sin\theta \cos\theta \tan\theta$ equals in the simplest form
 (A) $\sin^2\theta$ (B) $\cos^2\theta$ (C) $\tan^2\theta$ (D) $1 - \sin^2\theta$
- ③ $\sin(90^\circ - \theta) \csc(90^\circ - \theta)$ equals : in the simplest form
 (A) 1 (B) $\sin^2\theta$ (C) $\cos^2\theta$ (D) $\sin\theta \cos\theta$
- ④ $\frac{1 - \cos^2\beta}{\sin^2\beta - 1}$ equals : in the simplest form
 (A) $-\tan^2\beta$ (B) $-\cot^2\beta$ (C) $\tan^2\beta$ (D) $\cot^2\beta$

Second: Answer the following questions

- ⑤ Prove the validity of the following identities:
- (A) $\tan\mu + \cot\mu = \sec\mu \csc\mu$ (B) $\csc\alpha - \sin\alpha = \cos\alpha \cot\alpha$
- (C) $\cot^2\mu - \cos^2\mu = \cot^2\mu \cos^2\mu$ (D) $\tan^2\alpha - \sin^2\alpha = \tan^2\alpha \sin^2\alpha$
- (E) $\sin^2\alpha + \tan^2\alpha \sin^2\alpha = \tan^2\alpha$ (F) $\sin(90^\circ - \mu) \cos\mu = 1 - \sin^2\mu$



⑥ Prove the validity of the following identities:

(A) $\frac{\csc \theta}{\cos \theta} (1 - \sin^2 \theta) = \cot \theta$

(B) $\frac{1}{\sin^2(90^\circ - \theta)} - \tan^2 \theta = 1$

(C) $\frac{1}{1 + \tan^2 \alpha} - \frac{1}{1 + \tan^2 \beta} = \cos^2 \alpha - \cos^2 \beta$

(D) $\frac{1 + \tan^2 \theta}{\sec^4 \theta} = 1 - \sin^2 \theta$

(E) $(\sec \phi - \tan \phi)^2 = \frac{1 - \sin \phi}{1 + \sin \phi}$

(F) $\frac{1}{1 + \cot \theta} = \frac{\tan \theta}{1 + \tan \theta}$

(G) $\frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos^2 \theta + \cos \theta \sin^2 \theta} = \csc \theta - \sec \theta$

Important basic skills:

① $x^2 - y^2 = (x - y)(x + y)$

② $x^4 - y^4 = (x^2 - y^2)(x^2 + y^2)$

③ $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

④ $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

① $\frac{A}{x} \pm \frac{B}{x} = \frac{A \pm B}{x}$

② $\frac{A}{x} \pm \frac{B}{y} = \frac{Ay \pm Bx}{xy}$

③ $\frac{A}{x} \pm \frac{B}{xy} = \frac{Ay \pm B}{xy}$

① $\frac{x}{\frac{a}{b}} = \frac{x b}{a}$

② $\frac{\frac{x}{a}}{b} = \frac{x}{a b}$

③ $\frac{\frac{x}{y}}{\frac{a}{b}} = \frac{x b}{y a}$

① $(x + y)^2 = x^2 + 2xy + y^2$

② $(x - y)^2 = x^2 - 2xy + y^2$

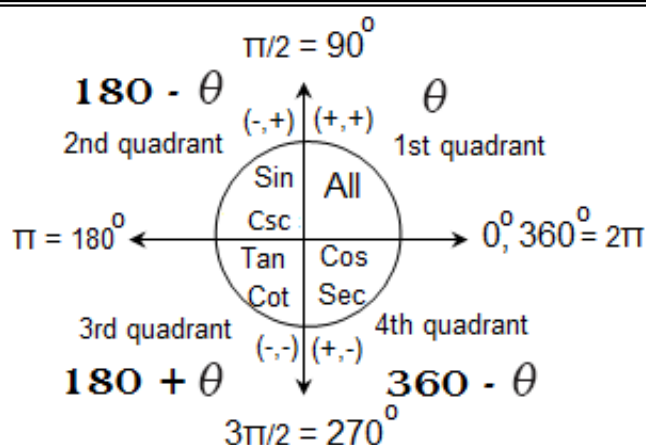
90°
270°

Shift

sin $\Leftrightarrow$ cos
csc $\Leftrightarrow$ sec
tan $\Leftrightarrow$ cot

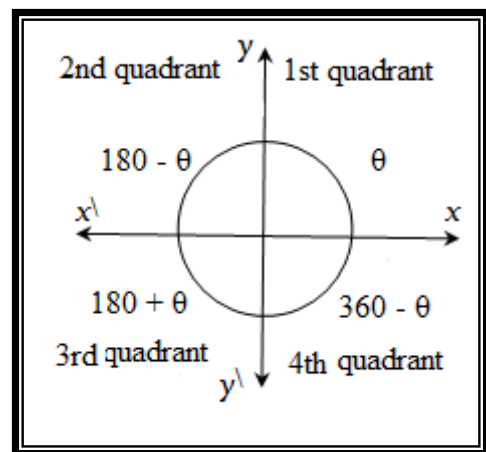
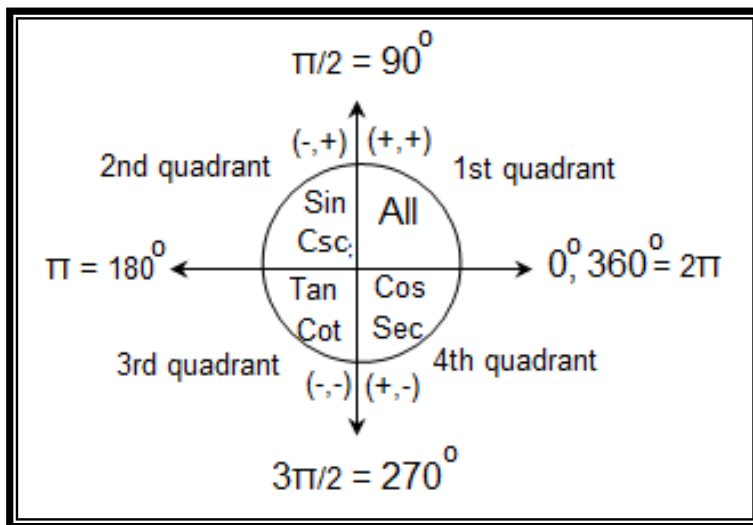
180°
360°

No change



**Lesson (5-2)****Solving trigonometric Equations****Lesson objectives****Related Links**

- Find the general solution for the trigonometric equation.
- Solve equations in the interval $[0, 2\pi[$

**Example 1**

Find the S.S. of: $2 \cos x + \sqrt{2} = 0$ $x \in] 0, 2 \pi [$

Solution

$$\therefore 2 \cos x + \sqrt{2} = 0$$

$$\therefore 2 \cos x = -\sqrt{2}$$

$$\therefore \cos x = -\frac{\sqrt{2}}{2}$$

$\therefore x$ lies in 2nd or 3rd quadrant

$$\text{Let } \cos \theta = \frac{\sqrt{2}}{2}$$

$$\therefore \theta = 45^\circ$$

x in 2nd quadrant

x in 3rd quadrant

$$\therefore x = 180^\circ - 45^\circ = 135^\circ$$

$$\therefore x = 180^\circ + 45^\circ = 225^\circ$$

$$\therefore \text{The S.S.} = \{135^\circ, 225^\circ\}$$

**Example 2**

Find the S.S. of: $2 \cos \alpha + 1 = 0$ $\alpha \in] \frac{\pi}{2}, \pi [$

Solution

$$\therefore 2 \cos \alpha + 1 = 0$$

$$\therefore 2 \cos \alpha = -1$$

$$\therefore \cos \alpha = -\frac{1}{2}$$

$\therefore x$ lies in 2nd quadrant

$$\text{Let } \cos \theta = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$

$\therefore \alpha$ in 2nd quadrant

$$\therefore \alpha = 180^\circ - 60^\circ = 120^\circ$$

$\therefore$ The S.S. = $\{120^\circ\}$

Example 3

Find the S.S. of: $2 \sin^2 \alpha = \sin \alpha$ $\alpha \in] 0, 2\pi [$

Solution

$$\therefore 2 \sin^2 \alpha - \sin \alpha = 0$$

$$\therefore \sin \alpha (2 \sin \alpha - 1) = 0$$

$$\therefore \sin \alpha = 0$$

$$\therefore 2 \sin \alpha - 1 = 0$$

$$\therefore \sin \alpha = 0$$

$$\therefore \sin \alpha = \frac{1}{2}$$

$\therefore \alpha$ is quadrantal angle

$\therefore \alpha$ lies in 1st or 2nd quadrant

$$\therefore \alpha = 0^\circ \text{ or } 360^\circ \quad (\text{refused})$$

$$\text{Let } \sin \theta = \frac{1}{2} \quad \therefore \theta = 30^\circ$$

Or

$$\therefore \alpha = 180^\circ$$

α in 1st quadrant

α in 2nd quadrant

$$\therefore \alpha = 30^\circ$$

$$\therefore \alpha = 180^\circ - 30^\circ = 150^\circ$$

$\therefore$ The S.S. = $\{30^\circ, 150^\circ, 180^\circ\}$

**Example 4**

Find the S.S. of: $3 \sec^2 \theta = 4$ $\theta \in] 0, 2\pi [$

Solution

$$\therefore 3 \sec^2 \theta = 4 \qquad \therefore \sec^2 \theta = \frac{4}{3} \qquad \therefore \cos^2 \theta = \frac{3}{4}$$

$$\therefore \cos \theta = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2}$$

$$\therefore \cos \theta = \frac{\sqrt{3}}{2}$$

$\therefore \theta$ lies in 1st or 4th quadrant

$$\text{Let } \cos x = \frac{\sqrt{3}}{2} \qquad \therefore x = 30^\circ$$

θ in 1st quadrant

$$\therefore \theta = 30^\circ$$

θ in 4th quadrant

$$\therefore \theta = 360^\circ - 30^\circ$$

$$\therefore \theta = 330^\circ$$

$$\therefore \cos \theta = -\frac{\sqrt{3}}{2}$$

$\therefore \theta$ lies in 2nd or 3rd quadrant

$$\text{Let } \cos x = \frac{\sqrt{3}}{2} \qquad \therefore x = 30^\circ$$

θ in 2nd quadrant

$$\therefore \theta = 180^\circ - 30^\circ$$

$$\therefore \theta = 150^\circ$$

θ in 3rd quadrant

$$\therefore \theta = 180^\circ + 30^\circ$$

$$\therefore \theta = 210^\circ$$

$$\therefore \text{The S.S.} = \{30^\circ, 150^\circ, 210^\circ, 330^\circ\}$$

Example 5

Find the S.S. of: $2 \cos^2 x - 5 \cos x - 3 = 0$ $x \in] 0, 2\pi [$

Solution

$$\therefore 2 \cos^2 x - 5 \cos x - 3 = 0$$

$$\therefore \cos x = 3 \quad (\text{refused})$$

$$\therefore \cos x = -\frac{1}{2}$$

$\therefore x$ lies in 2nd or 3rd quadrant

$$\text{Let } \cos \theta = \frac{1}{2} \qquad \therefore \theta = 60^\circ$$

x in 2nd quadrant

$$\therefore x = 180^\circ - 60^\circ = 120^\circ$$

x in 3rd quadrant

$$\therefore x = 180^\circ + 60^\circ = 240^\circ$$

$$\therefore \text{The S.S.} = \{120^\circ, 240^\circ\}$$

**Example 6**

Find the S.S. of: $\sin^2 \theta - \cos^2 \theta = \frac{1}{2}$ $\theta \in]0, 2\pi [$

Solution

$$\therefore \sin^2 \theta - \cos^2 \theta = \frac{1}{2}$$

$$\therefore \cos^2 \theta = 1 - \sin^2 \theta$$

$$\therefore \sin^2 \theta - (1 - \sin^2 \theta) = \frac{1}{2}$$

$$\therefore \sin^2 \theta - 1 + \sin^2 \theta = \frac{1}{2}$$

$$\therefore 2 \sin^2 \theta = \frac{3}{2}$$

$$\therefore \sin^2 \theta = \frac{3}{4}$$

$$\therefore \sin \theta = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2}$$

$$\therefore \sin \theta = \frac{\sqrt{3}}{2}$$

$\therefore \theta$ lies in 1st or 2nd quadrant

$$\text{Let } \sin x = \frac{\sqrt{3}}{2} \quad \therefore x = 60^\circ$$

θ in 1st quadrant

$$\therefore \theta = 60^\circ$$

θ in 2nd quadrant

$$\therefore \theta = 180^\circ - 60^\circ$$

$$\therefore \theta = 120^\circ$$

$$\therefore \sin \theta = -\frac{\sqrt{3}}{2}$$

$\therefore \theta$ lies in 3rd or 4th quadrant

$$\text{Let } \sin x = \frac{\sqrt{3}}{2} \quad \therefore x = 60^\circ$$

θ in 3rd quadrant

$$\therefore \theta = 180^\circ + 60^\circ$$

$$\therefore \theta = 240^\circ$$

θ in 4th quadrant

$$\therefore \theta = 360^\circ - 60^\circ$$

$$\therefore \theta = 300^\circ$$

$\therefore$ The S.S. = $\{60^\circ, 120^\circ, 240^\circ, 300^\circ\}$

**PRACTICE****Choose the correct answer**

Q (1): Find the S.S. satisfying $6 \cos^2 \theta - 7 \cos \theta - 5 = 0$ where $0^\circ \leq \theta < 360^\circ$

- Ⓐ $\{120^\circ, 240^\circ\}$
- Ⓑ $\{120^\circ, 300^\circ\}$
- Ⓒ $\{60^\circ, 300^\circ\}$
- Ⓓ $\{60^\circ, 240^\circ\}$

Q (2): Find the set of values satisfying $6 \cos^2 \theta - \cos \theta - 1 = 0$ where $0^\circ \leq \theta < 360^\circ$.
Give the answers to the nearest minute.

- Ⓐ $\{60^\circ, 109^\circ 28', 300^\circ, 250^\circ 32'\}$
- Ⓑ $\{120^\circ, 300^\circ, 109^\circ 28', 289^\circ 28'\}$
- Ⓒ $\{120^\circ, 240^\circ, 70^\circ 32', 289^\circ 28'\}$
- Ⓓ $\{60^\circ, 300^\circ, 70^\circ 32', 250^\circ 32'\}$

Q (3): Find all the possible general solutions of $2 \cos^2 \theta - \sqrt{2} \cos \theta = 0$.

- Ⓐ $\frac{\pi}{2} + n\pi, -\frac{\pi}{2} + n\pi, \frac{\pi}{4} + 2n\pi, -\frac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$
- Ⓑ $\frac{\pi}{2} + 2n\pi, -\frac{\pi}{2} + 2n\pi, \frac{\pi}{4} + 2n\pi, -\frac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$
- Ⓒ $\frac{\pi}{2} + n\pi, \frac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$
- Ⓓ $\frac{\pi}{2} + n\pi, -\frac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$



Q (4): Find all the possible general solutions of $2 \cos^2 \theta - \sqrt{3} \cos \theta = 0$

- (a) $\frac{\pi}{2} + n\pi, -\frac{\pi}{2} + n\pi, \frac{\pi}{6} + 2n\pi, -\frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}$
- (b) $\frac{\pi}{2} + 2n\pi, -\frac{\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, -\frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}$
- (c) $\frac{\pi}{2} + n\pi, \frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}$
- (d) $\frac{\pi}{2} + n\pi, -\frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}$

Q (5): Find the S.S. which satisfy: $2 \sin^2 \theta - \sqrt{2} \cos \theta - 2 = 0, 180^\circ \leq \theta < 360^\circ$

- (a) $\{45^\circ, 135^\circ\}$
- (b) $\{135^\circ, 225^\circ\}$
- (c) $\{135^\circ, 315^\circ\}$
- (d) $\{225^\circ, 315^\circ\}$

Q (6): Find the S.S. which satisfy: $\tan^2 \theta + \tan \theta = 0, 0^\circ \leq \theta < 180^\circ$

- (a) $\{135^\circ, 315^\circ, 0^\circ, 180^\circ\}$
- (b) $\{135^\circ, 225^\circ, 0^\circ, 180^\circ\}$
- (c) $\{135^\circ, 45^\circ, 90^\circ, 270^\circ\}$
- (d) $\{45^\circ, 135^\circ, 0^\circ, 90^\circ\}$

Q (7): Find the possible satisfy: $\sin \theta + \cos \theta = 1, 0^\circ \leq \theta < 180^\circ$

- (a) $0^\circ, 45^\circ$
- (b) $45^\circ, 90^\circ$
- (c) $0^\circ, 90^\circ$
- (d) $90^\circ, 180^\circ$



Home Work

First : Complete each of the following

- ① The general solution of the equation $\cos \theta = 1$ for all values of θ is
- ② The general solution of the equation $\sin \theta = 1$ where $\theta \in [\pi, 2\pi[$ is
- ③ The general solution of the equation $\sin \theta = \cos \theta$ for all values of θ is
- ④ The solution set of the equation $\cot \theta = \sqrt{3}$ where $\theta \in [\pi, 2\pi[$ is

Second : Multiple choice

- ⑤ If $0^\circ \leq \theta < 360^\circ$, $\sin \theta + 1 = 0$, then θ equals
 (A) 0° (B) 90° (C) 180° (D) 270°
- ⑥ If $0^\circ \leq \theta < 360^\circ$, $\cos \theta + 1 = 0$, then θ equals
 (A) 90° (B) 180° (C) 270° (D) 360°
- ⑦ If $0^\circ \leq \theta < 180^\circ$, $\sqrt{3} \tan \theta - 1 = 0$, then θ equals
 (A) 30° (B) 60° (C) 120° (D) 150°
- ⑧ If $180^\circ \leq \theta < 360^\circ$, $2 \cos \theta + 1 = 0$, then θ equals
 (A) 210° (B) 240° (C) 300° (D) 330°

Third: Answer the following questions

- ⑨ Find the general solution for each of the following equations.

(A) $\sin \theta = \frac{1}{2}$

(B) $2 \cos \theta - \sqrt{3} = 0$

(C) $\sqrt{3} \tan \theta - 1 = 0$

- ⑩ Solve each of the following equations in the interval $[0, \frac{3\pi}{2}]$:

(A) $\tan^2 \theta - \tan \theta = 0$

(B) $2 \sin \theta \cos \theta - \cos \theta = 0$

(C) $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

**Lesson (5-3)****Solving the right-angled triangle****Lesson objectives****Related Links**

- ✚ Solve the right angled triangle given lengths of two sides.
- ✚ Solve the right angled triangle given length of a side and measure of its acute angles.

$$\bullet \sin \theta = \frac{opp}{hyp}$$

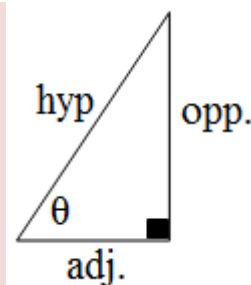
$$\bullet \cos \theta = \frac{adj}{hyp}$$

$$\bullet \tan \theta = \frac{opp}{adj}$$

$$\bullet \operatorname{cosec} \theta = \frac{hyp}{opp}$$

$$\bullet \sec \theta = \frac{hyp}{adj}$$

$$\bullet \cot \theta = \frac{adj}{opp}$$

**Example ①**

Solve: ΔABC where $m(\angle B) = 90^\circ$ in which $AB = 3$ cm, $BC = 4$ cm.

Solution

In ΔABC :

$m(\angle B) = 90^\circ$, $AB = 3$ cm , $BC = 4$ cm

$$AC = \sqrt{3^2 + 4^2} = 5 \text{ cm}$$

$$\therefore \sin A = \frac{4}{5}$$

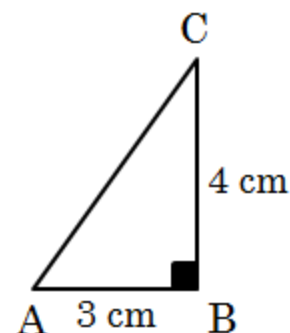
USE **SHIFT** → **SIN**

$$\therefore m(\angle A) = 53^\circ 7' 48''$$

$$\therefore \tan C = \frac{3}{4}$$

USE **SHIFT** → **TAN**

$$\therefore m(\angle C) = 36^\circ 52' 12''$$



**Example 2**

Solve: $\triangle ABC$ where $m(\angle C) = 90^\circ$ in which $AC = 8$ cm, $AB = 12.5$ cm.

Solution

In $\triangle ABC$:

$$m(\angle C) = 90^\circ, \quad AC = 8 \text{ cm}, \quad AB = 12.5 \text{ cm}$$

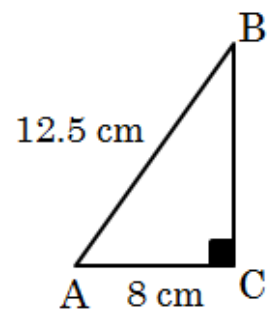
$$BC = \sqrt{12.5^2 - 8^2} \approx 9.6 \text{ cm}$$

$$\therefore \cos A = \frac{8}{12.5} \quad \text{USE } \boxed{\text{SHIFT}} \rightarrow \boxed{\text{COS}}$$

$$\therefore m(\angle A) = 50^\circ 10' 29''$$

$$\therefore \cos B = \frac{9.6}{12.5} \quad \text{USE } \boxed{\text{SHIFT}} \rightarrow \boxed{\text{COS}}$$

$$\therefore m(\angle B) = 39^\circ 49' 31''$$

**Example 3**

Solve: $\triangle ABC$ where $m(\angle B) = 90^\circ$ in which $AB = 8.6$ cm, $m(\angle C) = 41^\circ 18'$

Solution

In $\triangle ABC$:

$$m(\angle B) = 90^\circ, \quad AB = 8.6 \text{ cm}, \quad m(\angle C) = 41^\circ 18'$$

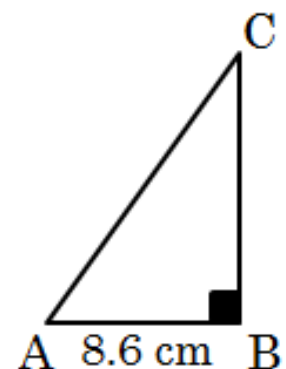
$$m(\angle A) = 180^\circ - (90^\circ + 41^\circ 18') = 48^\circ 42'$$

$$\therefore \tan A = \frac{BC}{AB}$$

$$\therefore \tan 48^\circ 42' = \frac{BC}{8.6}$$

$$\therefore BC = 8.6 \times \tan 48^\circ 42' \approx 9.8 \text{ cm}$$

$$\therefore AC = \sqrt{8.6^2 + 9.8^2} \approx 13 \text{ cm}$$



**Example 4**

Solve: $\triangle ABC$ where $m(\angle B) = 90^\circ$ in which $AB = 15$ cm, $m(\angle C) = 35^\circ 18'$

Solution

In $\triangle ABC$:

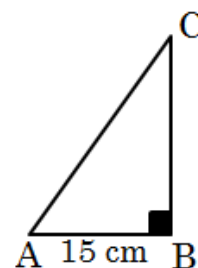
$$m(\angle B) = 90^\circ, \quad AB = 15 \text{ cm}, \quad m(\angle C) = 35^\circ 18'$$

$$m(\angle A) = 180^\circ - (90^\circ + 35^\circ 18') = 54^\circ 42'$$

$$\therefore \tan C = \frac{AB}{BC}$$

$$\therefore \tan 35^\circ 18' = \frac{15}{BC}$$

$$\therefore BC = \frac{15}{\tan 35^\circ 18'} \simeq 21 \text{ cm} \quad \therefore AC = \sqrt{21^2 + 15^2} \simeq 26 \text{ cm}$$

**Example 5**

ABC is an acute-angled triangle. $\overline{AD} \perp \overline{BC}$ where $\overline{AD} \cap \overline{BC} = \{D\}$.

If $AD = 8.4$ cm, $m(\angle B) = 63^\circ 28'$ and $m(\angle C) = 35^\circ 41'$ Find the length of $\overline{BC}$.

Solution

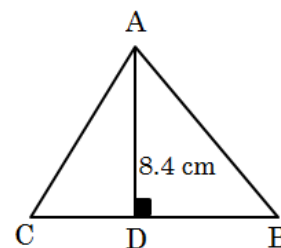
In $\triangle ABD$:

$$m(\angle ADB) = 90^\circ, \quad m(\angle B) = 63^\circ 28'$$

$BD = ??$

$$\therefore \tan B = \frac{AD}{BD} \quad \therefore \tan 63^\circ 28' = \frac{8.4}{BD}$$

$$\therefore BD = \frac{8.4}{\tan 63^\circ 28'} \simeq 4.2 \text{ cm}$$



In $\triangle ACD$:

$$m(\angle ADC) = 90^\circ, \quad m(\angle C) = 35^\circ 41'$$

$CD = ??$

$$\therefore \tan C = \frac{AD}{CD} \quad \therefore \tan 35^\circ 41' = \frac{8.4}{CD}$$

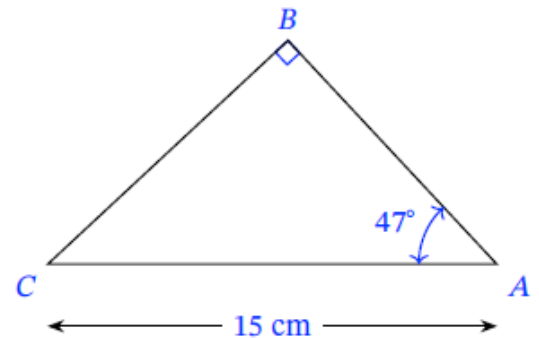
$$\therefore CD = \frac{8.4}{\tan 35^\circ 41'} \simeq 11.7 \text{ cm}$$

$$\therefore BC = 4.2 + 11.7 = 15.9 \simeq 16 \text{ cm}$$

**PRACTICE****Choose the correct answer**

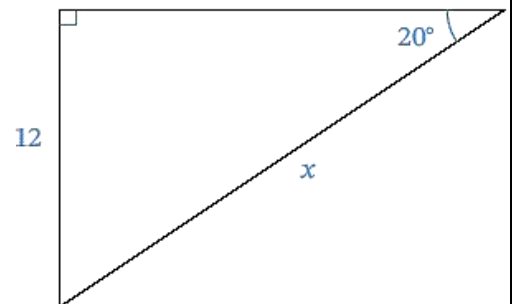
Q (1): Find the length of $\overline{BC}$ giving the answer to two decimal places.

- Ⓐ 10.23 cm
- Ⓑ 13.99 cm
- Ⓒ 16.09 cm
- Ⓓ 10.97 cm



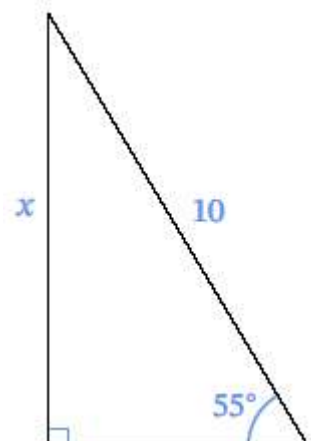
Q (2): Find x to two decimal places.

- Ⓐ 32.97
- Ⓑ 12.77
- Ⓒ 15.43
- Ⓓ 35.09



Q (3): Find x to two decimal places.

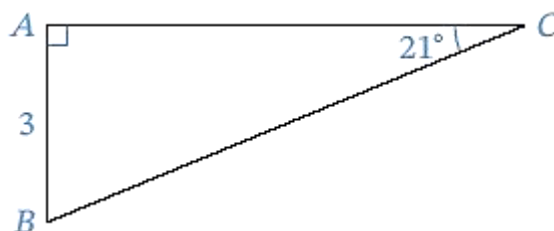
- Ⓐ 8.91
- Ⓑ 7.07
- Ⓒ 5.74
- Ⓓ 8.19





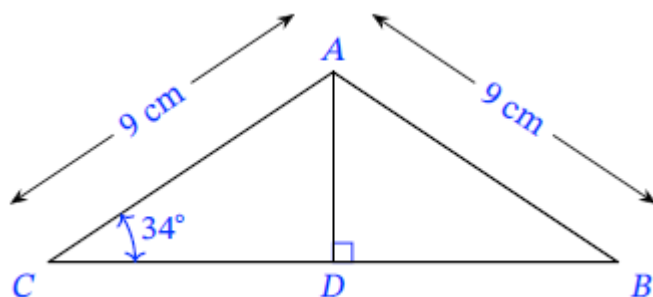
Q (4): Given the following figure, find the lengths of $\overline{AC}$ and $\overline{BC}$ and the measure of $\angle ABC$ in degrees. Give your answers to two decimal places.

- Ⓐ $AC = 7.82, BC = 8.37, m(\angle ABC) = 69^\circ$
- Ⓑ $AC = 7.82, BC = 8.37, m(\angle ABC) = 60^\circ$
- Ⓒ $AC = 7.32, BC = 7.91, m(\angle ABC) = 69^\circ$
- Ⓓ $AC = 7.32, BC = 7.91, m(\angle ABC) = 60^\circ$



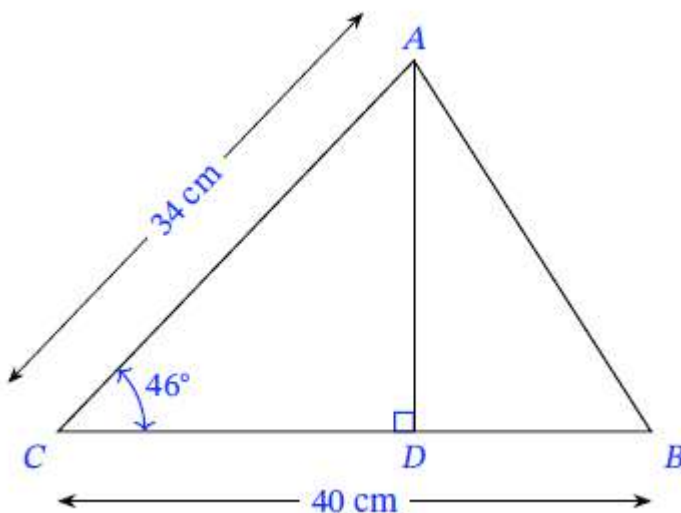
Q (5): ABC is an isosceles Δ where $AB = AC = 9$ cm, $\overline{AD} \perp \overline{BC}$ and $m(\angle C) = 34^\circ$. Find the length of $\overline{BC}$ giving the answer to one decimal place.

- Ⓐ 10.1 cm
- Ⓑ 14.9 cm
- Ⓒ 12.1 cm
- Ⓓ 7.5 cm



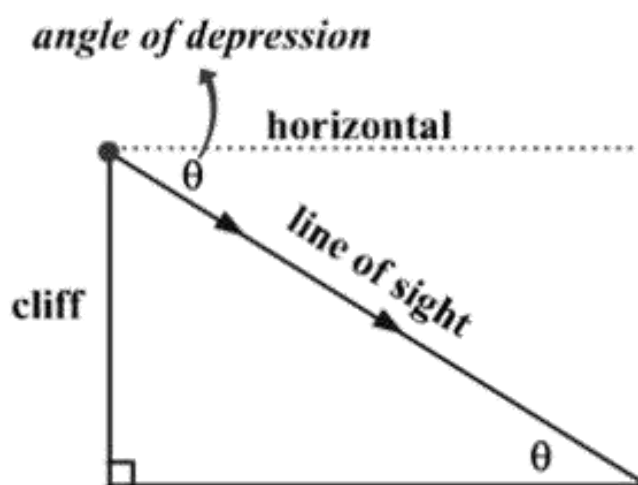
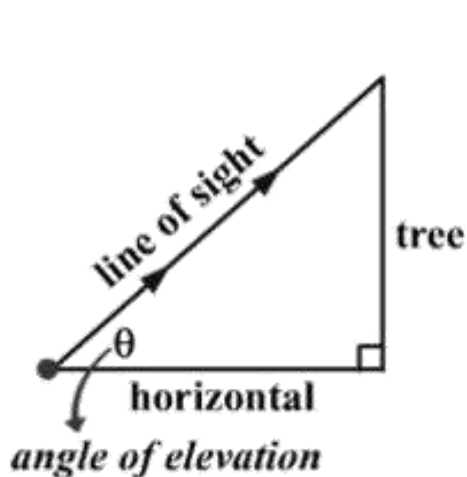
Q (6): ABC is a triangle where $AC = 34$ cm, $BC = 40$ cm, and $m(\angle C) = 46^\circ$. Point D lies on $\overline{BC}$ where $\overline{AD} \perp \overline{BC}$. Find the height of the triangle drawn from point B to $\overleftrightarrow{AC}$ giving the answer to two decimal places.

- Ⓐ 28.77 cm
- Ⓑ 41.42 cm
- Ⓒ 27.79 cm
- Ⓓ 24.46 cm



**Lesson (5-4)****Angles of elevation and depression****Lesson objectives****Related Links**

- Identify Concept of angles of elevation and depression.
- Use the right angled triangle to solve problems including angles of elevation and depression.

**Example ①**

A man find that the measure of the angle of elevation of the top of a house at a point 50 meters from the base is $38^\circ 26'$ Find the height of the house.

Solution

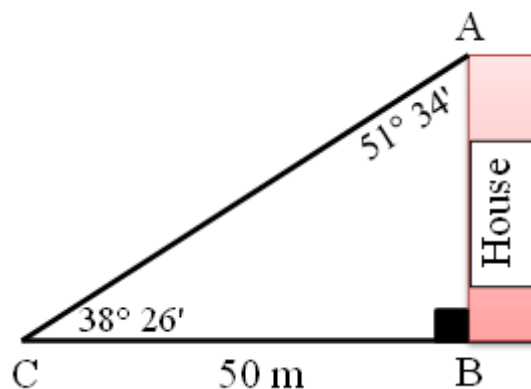
In ΔABC :

$$\because m(\angle B) = 90^\circ$$

$$\because \tan C = \frac{AB}{BC} \quad \therefore \tan(38^\circ 26') = \frac{AB}{50}$$

$$\therefore AB = 50 \times \tan(38^\circ 26') = 39.67 \approx 40 \text{ m}$$

$$\therefore \text{The height of the house} = 40 \text{ m}$$



**Example 2**

From the top of a tower of height 50 m., the angle of depression of a body in the same horizontal plane with its base $23^\circ 24'$ find the distance between this body and the base of the tower to the nearest meter.

Solution

In ΔABC :

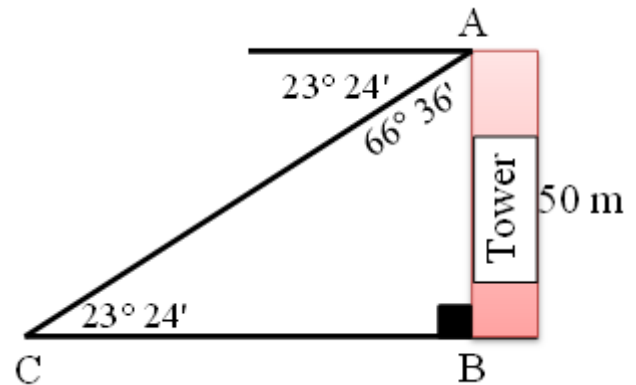
$$\because m(\angle B) = 90^\circ$$

$$\because \tan C = \frac{AB}{BC}$$

$$\therefore \tan(23^\circ 24') = \frac{50}{BC}$$

$$\therefore BC = \frac{50}{\tan(23^\circ 24')} = 115.5 \approx 116 \text{ m}$$

$\therefore$ The distance between this body and the base of the tower = 116 m

**Example 3**

A man of height 1.5 m. was standing on the ground at a point which is 10 m. far from a flagpole. He found that the angle of elevation of the top of the flagpole is $40^\circ 22'$. Find the height of the flagpole.

Solution

In ΔABC :

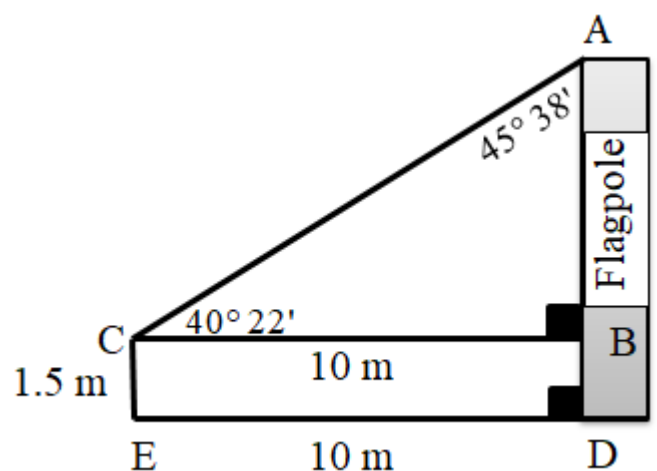
$$\because m(\angle B) = 90^\circ$$

$$\because \tan C = \frac{AB}{BC}$$

$$\therefore \tan(40^\circ 22') = \frac{AB}{10}$$

$$\therefore AB = 10 \times \tan(40^\circ 22') = 9.78 \text{ m}$$

$\therefore$ The height of the flagpole = $9.78 + 1.5 \approx 11 \text{ m}$



**Example 4**

From the top of a tower of 50 m. high, the measures of the two angles of depression of two sailboats are $32^\circ 10'$ and $49^\circ 30'$. Find the distance between the two sailboats.

Solution

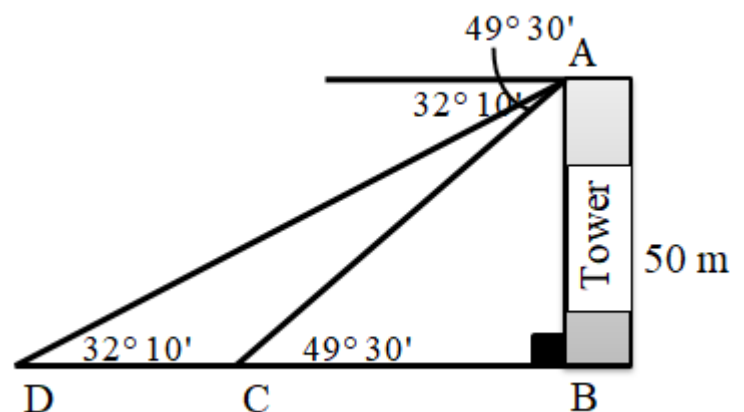
In ΔABC :

$$\because m(\angle B) = 90^\circ$$

$$\because \tan C = \frac{AB}{BC}$$

$$\therefore \tan(49^\circ 30') = \frac{50}{BC}$$

$$\therefore BC = \frac{50}{\tan(49^\circ 30')} = 42.7 \text{ m.}$$



In ΔABD :

$$\because m(\angle B) = 90^\circ$$

$$\because \tan D = \frac{AB}{BD}$$

$$\therefore \tan(32^\circ 10') = \frac{50}{BD}$$

$$\therefore BD = \frac{50}{\tan(32^\circ 10')} = 79.5 \text{ m.}$$

$\therefore$ The distance between the two boats = $79.5 - 42.7 = 36.8 \approx 37 \text{ m}$

**Example 5**

From the top of a rock of height 100 m, a man found that the measure of the angles of depression of two boats are $65^\circ 18'$ and $43^\circ 52'$. Given that the two boats are in two different ways from the bottom of the rock. Find to the nearest meter the distance between the two boats.

Solution

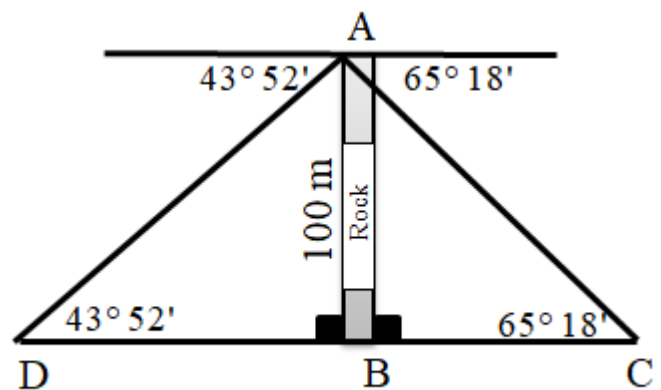
In $\triangle ABC$:

$$\because m(\angle B) = 90^\circ$$

$$\because \tan C = \frac{AB}{BC}$$

$$\therefore \tan(65^\circ 18') = \frac{100}{BC}$$

$$\therefore BC = \frac{100}{\tan(65^\circ 18')} = 46 \text{ m.}$$



In $\triangle ABD$:

$$\because m(\angle B) = 90^\circ$$

$$\because \tan D = \frac{AB}{BD}$$

$$\therefore \tan(43^\circ 52') = \frac{100}{BD}$$

$$\therefore BD = \frac{100}{\tan(43^\circ 52')} = 104 \text{ m.}$$

$\therefore$ The distance between the two boats = $104 + 46 = 150 \text{ m}$

**Example 6**

The angle of elevations of the top and the base of the flag-pole placed on the rock was measures to be $36^\circ 20'$ and $47^\circ 30'$ if the distance between the base of the rock and the point of observation is 50 m. find the height of the flag.

Solution

In $\triangle DBC$:

$$\because m(\angle C) = 90^\circ$$

$$\because \tan D = \frac{BC}{DC}$$

$$\therefore \tan(36^\circ 20') = \frac{BC}{50}$$

$$\therefore BC = 50 \times \tan(36^\circ 20') = 36.77 \text{ m}$$

In $\triangle ACD$:

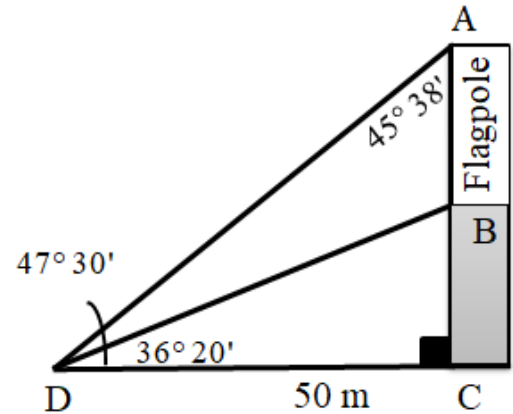
$$\because m(\angle C) = 90^\circ$$

$$\because \tan D = \frac{AC}{DC}$$

$$\therefore \tan(47^\circ 30') = \frac{AC}{50}$$

$$\therefore AC = 50 \times \tan(47^\circ 30') = 54.56 \text{ m}$$

$$\therefore \text{The height of the flag-pole} = 54.56 - 36.77 = 17.79 \approx 18 \text{ m}$$



**Example 7**

From a point on the ground a man found the elevation angle of the top of the tower was 45° when he walked towards the base of the tower the angle of elevation become 54° if the height of the tower was 100 m. find the distance covered by the man.

Solution

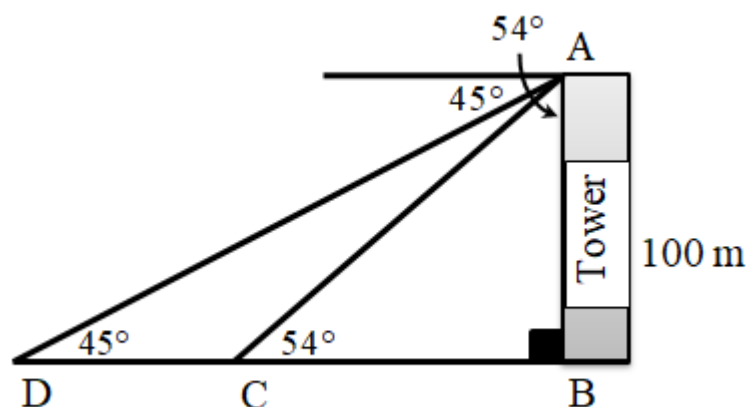
In ΔABC :

$$\because m(\angle B) = 90^\circ$$

$$\because \tan C = \frac{AB}{BC}$$

$$\therefore \tan 54^\circ = \frac{100}{BC}$$

$$\therefore BC = \frac{100}{\tan 54^\circ} = 72.65 \text{ m.}$$



In ΔABD :

$$\because m(\angle B) = 90^\circ$$

$$\because \tan D = \frac{AB}{BD}$$

$$\therefore \tan 45^\circ = \frac{100}{BD}$$

$$\therefore BD = \frac{100}{\tan 45^\circ} = 100 \text{ m.}$$

$$\therefore \text{The distance covered by the man} = 100 - 72.65 = 27.35 \approx 27 \text{ m}$$

**Example 8**

A boat was sailing towards a rock of 20 m high., it was found that the angle of elevation of the top of the tower was 15° after 10 minutes; it was found that the measure of the top of the tower was 38°

Find the velocity of the boat.

Solution

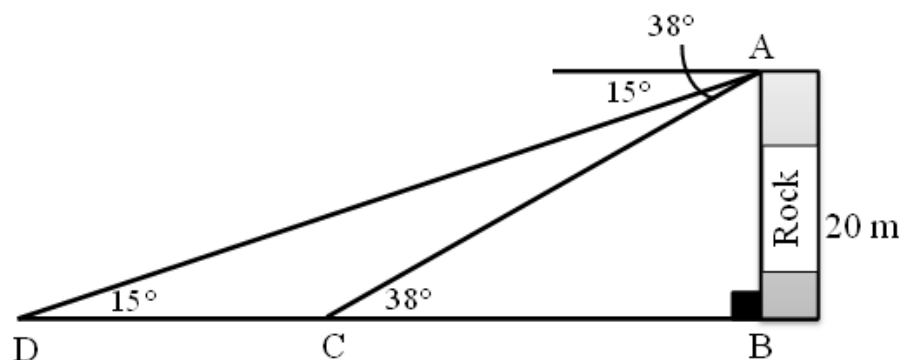
In $\triangle ABC$:

$$\therefore m(\angle B) = 90^\circ$$

$$\therefore \tan C = \frac{AB}{BC}$$

$$\therefore \tan 38^\circ = \frac{20}{BC}$$

$$\therefore BC = \frac{20}{\tan 38^\circ} = 25.6 \text{ m.}$$



In $\triangle ABD$:

$$\therefore m(\angle B) = 90^\circ$$

$$\therefore \tan D = \frac{AB}{BD}$$

$$\therefore \tan 15^\circ = \frac{20}{BD}$$

$$\therefore BD = \frac{20}{\tan 15^\circ} = 74.64 \text{ m.}$$

$$\therefore \text{The distance covered by the boat} = 74.64 - 25.6 = 49.04 \approx 49 \text{ m}$$

$$\therefore \text{The velocity of the boat} = \frac{d}{t} = \frac{49}{10} = 4.9 \text{ m/min.}$$

**PRACTICE****Choose the correct answer****Q (1):**

A building is 8 meters tall. The angle of elevation from the top of the building to the top of a tree is 44° and the angle of depression from the top of the building to the base of the tree is 58° . Find the distance between the base of the building and the base of the tree giving the answer to two decimal places.

- Ⓐ 10.49 m
- Ⓑ 9.43 m
- Ⓒ 6.10 m
- Ⓓ 5.18 m

Q (2):

The height of a lighthouse is 60 meters. The angles of elevation between two boats in the sea and the top of the lighthouse are 29° and 39° , respectively. Given that the two boats and the base of the lighthouse are collinear and that the boats are both on the same side of the lighthouse, find the distance between the two boats giving the answer to the nearest meter.

- Ⓐ 9 m
- Ⓑ 28 m
- Ⓒ 15 m
- Ⓓ 34 m



Q (3):

A ladder is leaning against a wall where the upper end is 4.5 m high from the ground. The angle of inclination of the ladder to the ground is 41° . Find the length of the ladder giving the answer to two decimal places.

- Ⓐ 5.96 m
- Ⓑ 0.15 m
- Ⓒ 6.86 m
- Ⓓ 5.18 m

Q (4):

Anthony and Victoria want to find the height of a statue. Anthony stands 5 meters from the base of the statue and measures the angle of elevation, from the ground, to be 65° . Victoria stands directly behind Anthony; she measures the angle of elevation, from the ground, to be 30° . They both calculate the same height for the statue. How far behind Anthony must Victoria be standing? Give your solution to two decimal places.

- Ⓐ 13.57 meters
- Ⓑ 18.57 meters
- Ⓒ 4.53 meters
- Ⓓ 10.72 meters



Q (5):

A flag is hung 22 meters up a flagpole. As the flag is raised, the angle of elevation from a point 21 meters away from the base of the flagpole to the flag is 74° . Find the increase in height of the flag giving the answer to two decimal places.

- Ⓐ 51.24 m
- Ⓑ 73.24 m
- Ⓒ 22.00 m
- Ⓓ 5.79 m

Q (6):

The angle of elevation to the top of a skyscraper from a 30 m high house is $58^\circ 42'$. The base of the house is 45 meters away from the base of the skyscraper. Find the height of the skyscraper giving the answer to two decimal places.

- Ⓐ 104.01 m
- Ⓑ 68.45 m
- Ⓒ 38.45 m
- Ⓓ 74.01 m



Q (7):

Two points on the ground lie collinearly on either side of a flagpole 29 meters tall. The angles of elevation from the two points to the top of the flagpole are $45^\circ 18'$ and $34^\circ 18'$. Find the distance between the two points giving the answer to one decimal place.

- Ⓐ 6.1 m
- Ⓑ 9.5 m
- Ⓒ 10.7 m
- Ⓓ 13.8 m

Q (8):

A man was standing 25 m away from the base of a tower with a flagpole at the top. He measured the angles of elevation of the top and the base of the flagpole to be 40° and 23° respectively. Find the height of the flagpole accurate to two decimal places.

- Ⓐ 29.10 m
- Ⓑ 6.30 m
- Ⓒ 10.37 m
- Ⓓ 3.86 m



Home Work

- ① The length of the thread of a kite is 42 metres, if the angle which the thread makes with the horizontal ground equals 63° , Find to the nearest metre the height of the kite from the surface of the ground.

- ② A man found that the angle of elevation of the top of a minaret 42m distance from its base was 52° . What is the height of the minaret to the nearest metre?

- ③ A mountain of height 1820 metres, It is observed from its top that the measure of depression angle of a point on the ground was 68° . What is the distance between the point and the observer to the nearest metre?

- ④ The upper end of a ladder rests on a vertical wall, it is 3.8m from the surface of the ground, the lower end rests on a horizontal ground. If the angle of inclination of the ladder to the ground is 64° . Find to the nearest hundredth each of:

A The distance between the lower end and the wall**B** The length of the ladder

- ⑤ From the top surface of a house 8 metres high, a person found that the elevation angle of the top of an opposite building was of measure 63° , and observed the depression angle of its base, it was 28° . Find the height of the building to the nearest metre.

- ⑥ If the measure of the elevation angle of the top of a minaret from the point 140 metres distance from its base was $26^\circ 46'$. What is the height of the minaret to the nearest metre? If it is measured from 110 metres distance from its base, find to the nearest minute, the measure of its elevation angle at that distance.

- ⑦ An observer measured the angle of elevation of a fixed ballon to be $\frac{\pi}{6}$, when he walked in a horizontal plane towards the ballon a distance 800 metres, he measures its angle of elevation to be $\frac{\pi}{4}$. Find the height of the ballon to the nearest metre.

- ⑧ A ship approaches a light house 50 metres high at a moment, it was found that the elevation angle of the top of the light house to be 0.11^{rad} , after 15 minutes, it was found again that its elevation angle to be 0.22^{rad} . Calculate the uniform velocity of the ship.

**Lesson (5-5)****Circular Sector****Lesson objectives****Related Links**

- Identify Concept of the circular sector.
- Find the area of the circular sector.

Remember that:

$$\bullet \theta^{\text{rad}} = \frac{\ell}{r} \quad \bullet L = \theta^{\text{rad}} \times r \quad \bullet r = \frac{\ell}{\theta^{\text{rad}}}$$

$$\bullet \theta^{\text{rad}} \longrightarrow \pi$$

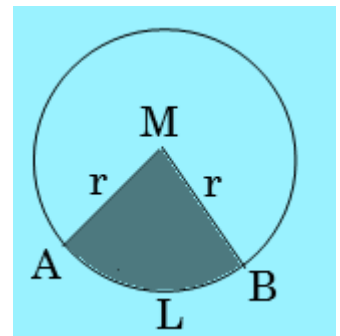
$$\bullet \theta^{\circ} \longrightarrow 180^{\circ}$$

1. The perimeter of the circular sector = $2r + \ell$

2. The area of the circular sector = $\frac{1}{2} \ell r$

$$\text{The area of the circular sector} = \frac{1}{2} \theta^{\text{rad}} r^2$$

$$\text{The area of the circular sector} = \frac{\theta^{\circ}}{360} \pi r^2$$

**Example 1**

Find the area of a sector where the measure of its angle is 144° and the length of its radius is 10.5 cm.

Solution

$$\text{Area} = \frac{\theta^{\circ}}{360^{\circ}} \times \pi r^2$$

$$= \frac{144^{\circ}}{360^{\circ}} \times \frac{22}{7} \times 10.5^2$$

$$= 138.6 \text{ cm}^2$$

$$\theta^{\circ} = 144^{\circ}$$

$$r = 10.5 \text{ cm.}$$

**Example 2**

The perimeter of a circular sector is 55 cm and the length of its radius is 12cm. Find the area of a sector.

Solution

$$\because \text{Perimeter} = 55 \text{ cm}$$

$$\therefore 2r + \ell = 55 \qquad \therefore 2(12) + \ell = 55$$

$$\therefore \ell = 55 - 24 = 31 \text{ cm}$$

$$\text{Area} = \frac{1}{2} \ell r = \frac{1}{2} \times 31 \times 12 = 186 \text{ cm}^2$$

$$P = 55 \text{ cm}$$

$$r = 12 \text{ cm.}$$

Example 3

The area of a circular sector is 270 cm^2 and the length of its radius is 15 cm, **find:**

- ① The length of the arc of this sector.
- ② The measure of its central angle in degrees and in radians.

Solution

$$\because \text{Area} = \frac{1}{2} \ell r$$

$$\therefore \frac{1}{2} \ell r = 270$$

$$\therefore \frac{1}{2} \ell = \frac{270}{15}$$

$$\therefore \frac{1}{2} \ell = 18$$

$$\therefore \ell = 18 \div \frac{1}{2}$$

$$\therefore \ell = 36 \text{ cm.}$$

$$\theta^{\text{rad}} = \frac{\ell}{r} = \frac{36}{15} = 2.4^{\text{rad}}$$

$$\theta^{\circ} = 2.4 \times 180^{\circ} \div \pi = 137^{\circ} 30' 36''$$

$$A = 270 \text{ cm}^2$$

$$r = 15 \text{ cm.}$$

**Example 4**

If the perimeter of circular sector is 12 cm and its area is 8 cm^2 . Find the measure of the central angle with degree & radian measure.

Solution

$$\because \text{Perimeter} = 12 \text{ cm.} \quad \therefore 2r + \ell = 12 \quad \text{①}$$

$$\because \text{Area} = 8 \text{ cm}^2 \quad \therefore \frac{1}{2} \ell r = 8 \quad \therefore \ell r = 16 \quad \text{②}$$

$$\text{From ①: } \ell = 12 - 2r \quad \text{③}$$

$$\text{From ②: } (12 - 2r)r = 16$$

$$\therefore 12r - 2r^2 - 16 = 0 \quad \therefore 2r^2 - 12r + 16 = 0$$

$$\therefore r = 4 \text{ cm.} \quad \text{or} \quad r = 2 \text{ cm.}$$

$$\therefore \ell = 4 \text{ cm.} \quad \text{or} \quad \ell = 8 \text{ cm.}$$

$$\theta^{\text{rad}} = \frac{\ell}{r} = \frac{4}{4} = 1^{\text{rad}} \quad \theta^{\text{rad}} = \frac{\ell}{r} = \frac{2}{8} = 0.25^{\text{rad}}$$

$$\theta^{\circ} = 1 \times 180^{\circ} \div \pi = 57^{\circ} 17' 45'' \quad \theta^{\circ} = 0.25 \times 180^{\circ} \div \pi = 14^{\circ} 19' 26''$$

$$P = 12 \text{ cm}$$

$$A = 8 \text{ cm}^2$$

Example 5

A circular sector its perimeter = 88 cm. the length of the arc is 32 cm. **Find** its area and the measure of its central angle.

Solution

$$\because \text{Perimeter} = 88 \text{ cm.} \quad \therefore 2r + \ell = 88$$

$$\therefore 2r + 32 = 88 \quad \therefore 2r = 56 \quad \therefore r = 28 \text{ cm.}$$

$$\because \text{Area} = \frac{1}{2} \ell r = \frac{1}{2} \times 32 \times 28 = 448 \text{ cm}^2$$

$$\theta^{\text{rad}} = \frac{\ell}{r} = \frac{32}{28} = \frac{8}{7}^{\text{rad}}$$

$$\theta^{\circ} = \frac{8}{7} \times 180^{\circ} \div \pi = 65^{\circ} 28' 51''$$

$$P = 88 \text{ cm}$$

$$\ell = 32 \text{ cm}$$

**Example 6**

The area of a circular sector is 25 cm^2 and the measure of its central angle $\theta^{\text{rad}} = 0.5^{\text{rad}}$. **Find** the length of the arc, and the length of radius.

Solution

$$\therefore \text{Area} = 25 \text{ cm}^2$$

$$\therefore \frac{1}{2} r^2 \theta^{\text{rad}} = 25$$

$$\therefore r^2 \theta^{\text{rad}} = 25 \div \frac{1}{2}$$

$$\therefore r^2 \theta^{\text{rad}} = 50$$

$$\therefore r^2 = 50 \div 0.5 = 100$$

$$\therefore r = \sqrt{100} = 10 \text{ cm.}$$

$$\therefore \theta^{\text{rad}} = \frac{\ell}{r}$$

$$\therefore 0.5 = \frac{\ell}{10}$$

$$\therefore \ell = 5 \text{ cm.}$$

$$A = 25 \text{ cm}^2$$

$$\theta^{\text{rad}} = 0.5^{\text{rad}}$$

Example 8

ABC is a right-angled triangle at B, in which $AB = 6 \text{ cm}$ and $BC = 8 \text{ cm}$.

Draw a circle arc with center A and with radius $\overline{AB}$ to intersect $\overline{AC}$ at D.

Find to the nearest cm^2 the area of the zone bounded by $\overline{BC}$ and $\overline{CD}$ and $\widehat{BD}$

Solution

In ΔABC :

$$\therefore \tan A = \frac{8}{6} = \frac{4}{3}$$

$$\therefore m(\angle A) = 53^\circ 7' 48''$$

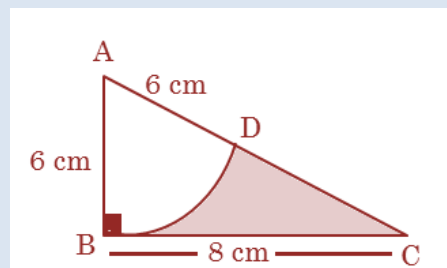
$$\therefore \theta^\circ = 53^\circ 7' 48''$$

$$\therefore \theta^{\text{rad}} = 53^\circ 7' 48'' \times \pi \div 180^\circ = 0.927^{\text{rad}}$$

$$\therefore \text{Area of require region} = \text{area of } \Delta ABC - \text{area of sector ADB}$$

$$= \frac{1}{2} b h - \frac{1}{2} r^2 \theta^{\text{rad}}$$

$$= \frac{1}{2} \times 8 \times 6 - \frac{1}{2} \times 6^2 \times 0.927 \simeq 7 \text{ cm}^2$$



**PRACTICE****Question 1**

The area of a circular sector is 122.5 and the length of the arc is 5 cm. Find the perimeter of the sector giving the answer to the nearest centimetre.

- ☐ A 103 cm
- ☐ B 54 cm
- ☐ C 29.5 cm
- ☐ D 490 cm

Question 2

An arc has a measure of $\frac{\pi}{3}$ radians and a radius of 5. Give the area of the sector, in terms of π , in its simplest form.

- ☐ A $\frac{25\pi}{3}$
- ☐ B $\frac{50\pi}{3}$
- ☐ C $\frac{25\pi}{6}$
- ☐ D $\frac{50\pi}{6}$
- ☐ E $\frac{5\pi}{2}$

Question 3

The area of a circular sector is $\frac{5}{6}$ of the circle it is in. Find the central angle giving the answer to the nearest degree.

- ☐ A 300°
- ☐ B 150°
- ☐ C 75°
- ☐ D 60°

**Question 4**

A circle with radius 27 cm has a sector cut from it. The perimeter of the sector is 102 cm. What is the area of the sector?

Question 5

The area of a circular sector is $\frac{1}{4}$ of the area of a circle, find, in radians, the central angle, correct to one decimal place.

Question 6

The area of a circular sector is $\frac{5}{8}$ of the area of the circle. The radius is 27 cm. Find the central angle in radians giving the answer to two decimal places and the perimeter of the sector giving the answer to the nearest centimeter.

- ☐ A θ rad = 3.93 rad, perimeter = 160 cm
- ☐ B θ rad = 1.96 rad, perimeter = 133 cm
- ☐ C θ rad = 3.93 rad, perimeter = 133 cm
- ☐ D θ rad = 3.93 rad, perimeter = 239 cm
- ☐ E θ rad = 1.96 rad, perimeter = 107 cm

Question 7

The area of a circular sector is $1\,888\text{ cm}^2$ and its central angle is 1.7 rad. Find the arc length of the sector giving the answer to the nearest centimetre.

- ☐ A 227 cm
- ☐ B 40 cm
- ☐ C 20 cm
- ☐ D 80 cm
- ☐ E 160 cm



Home Work

First : Complete each of the following

- ① Area of the circular sector, in which $l = 6$ cm, $r = 4$ cm equals
- ② Area of the circular sector, in which $r = 4$ cm, and its perimeter 20 cm equals cm^2 .
- ③ Perimeter of the circular sector in which , its area 24 cm^2 , length of its arc 8 cm equals

Second : Multiple choice

- ① Area of the circular sector in which , measure of its angle 1.2^{rad} and length of the radius of its circle 4 cm equals
 (A) 4.8 cm^2 (B) 9.6 cm^2 (C) 12.8 cm^2 (D) 19.6 cm^2
- ② Perimeter of the circular sector in which length of its arc 4 cm and length of the diameter of its circle 10 cm equals
 (A) 14 cm (B) 20 cm (C) 30 cm (D) 40 cm
- ③ Area of the circular sector in which, measure of its angle 120° , length of the radius of its circle 3 cm equals
 (A) $3 \pi \text{ cm}^2$ (B) $6 \pi \text{ cm}^2$ (C) $9 \pi \text{ cm}^2$ (D) $12 \pi \text{ cm}^2$
- ④ Area of the circular sector in which, its perimeter 12 cm, length of its arc 6 cm equals
 (A) 6 cm^2 (B) 9 cm^2 (C) 12 cm^2 (D) 18 cm^2
- ⑤ If the area of the circular sector equals 110 cm^2 , measure of its angle equals 2.2^{rad} , then length of the radius of its circle equals:
 (A) 2 cm (B) 5 cm (C) 10 cm (D) 20 cm

Third : Answer the following questions

- ① Find the area of the circular sector in which, length of the diameter of its circle is 20 cm and measure of its angle is 120°
- ② Find the area of the circular sector in which length of its arc is 16cm, and length of the radius of its circle is 9cm.
- ③ Find the area of the circular sector in which , length of its arc is 7cm and its perimeter equals 25cm.
- ④ **Agriculture:** Abasin flowers is in the shape of a circular sector, its area equals 48 m^2 , length of its arc equals 6m. Find its perimeter and the length of the radius of its circle.
- ⑤ The perimeter of a circular sector equals 24 cm and length of its arc equals 10 cm. Find the area of its surface.

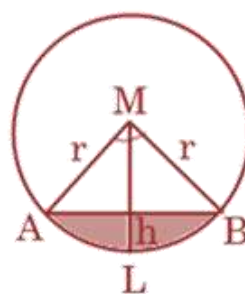
**Lesson (5–6)****Circular Segment****Lesson objectives****Related Links**

- ✚ Identify the circular segment.
- ✚ Find the area of the circular segment.

$$\text{Area of } \triangle ABC = \frac{1}{2} (AB) (AC) \sin A = \frac{1}{2} (BC) (AB) \sin B = \frac{1}{2} (AC) (BC) \sin C$$

Area of segment = Area of sector – Area of triangle

$$= \frac{1}{2} r^2 [\theta^{\text{rad}} - \sin \theta^{\circ}]$$

**Example ①**

Circular sector of radius 10 cm its Area = 100 cm². Find the area of segment which has the same arc.

Solution

$$\therefore \text{Area} = \frac{1}{2} \ell r$$

$$\therefore \frac{1}{2} \ell r = 100$$

$$\therefore \ell = \frac{100}{5} = 20 \text{ cm.}$$

$$\theta^{\text{rad}} = \frac{\ell}{r} = \frac{20}{10} = 2^{\text{rad}}$$

$$\theta^{\circ} = 2 \times 180^{\circ} \div \pi = 114^{\circ} 35' 30''$$

$$\therefore \text{Area of circular segment} = \frac{1}{2} r^2 [\theta^{\text{rad}} - \sin \theta^{\circ}]$$

$$= \frac{1}{2} \times 10^2 [2 - \sin 114^{\circ} 35' 30''] \approx 54.5 \text{ cm}^2$$

$$A = 100 \text{ cm}^2$$

$$r = 10 \text{ cm}$$

**Example 2**

Find the area of segment if: $r = 10$ cm and $\theta^\circ = 135^\circ$

Solution

$$\theta^{\text{rad}} = 135^\circ \times \pi \div 180^\circ = \frac{3}{4}\pi$$

$$\therefore \text{Area of circular segment} = \frac{1}{2} r^2 [\theta^{\text{rad}} - \sin \theta^\circ]$$

$$= \frac{1}{2} \times 10^2 \left[\frac{3}{4}\pi - \sin 135^\circ \right] \simeq 82.5 \text{ cm}^2$$

Example 3

$\overline{AB}$ is a chord in a circle whose center is M.

If $AB = 10$ cm & $m(\angle AMB) = 60^\circ$. Find the area of greater segment.

Solution

Const.: Draw $\overline{MC} \perp \overline{AB}$

In $\triangle AMC$:

$$\therefore m(\angle MCA) = 90^\circ, \quad m(\angle AMC) = 30^\circ$$

$$\therefore MA = 2 AC = 10 \text{ cm.} \quad \therefore r = 10 \text{ cm.}$$

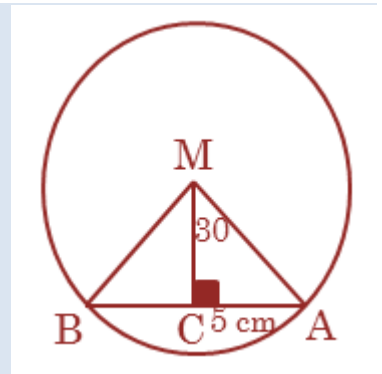
$$\theta^\circ = 60^\circ$$

$$\theta^{\text{rad}} = 60^\circ \times \pi \div 180^\circ = \frac{1}{3}\pi$$

$$\therefore \text{Area of greater segment} = \text{area of circle} - \text{area of smaller segment}$$

$$= \pi r^2 - \frac{1}{2} r^2 [\theta^{\text{rad}} - \sin \theta^\circ]$$

$$= \pi \times 10^2 - \frac{1}{2} \times 10^2 \left[\frac{1}{3}\pi - \sin 60^\circ \right] \simeq 305 \text{ cm}^2$$



**Example 4**

Two congruent circles, the length of the radius of each equals 12 cm. Each of them passes through the center of the other. Find the area of the common region between them.

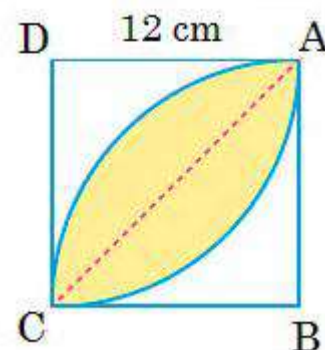
Solution

We draw $\overline{AC}$ which divides the shaded part into two equal segments in area where the central angle of each of them equals 90° and the radius of each of them equals 12 cm.

$$r = 12 \text{ cm.}$$

$$\theta^\circ = 90^\circ$$

$$\theta^{\text{rad}} = 90^\circ \times \pi \div 180^\circ = \frac{\pi}{2}$$



Area of the shaded part = $2 \times$ Area of the circular segment

$$\begin{aligned} &= 2 \times \frac{1}{2} r^2 [\theta^{\text{rad}} - \sin \theta^\circ] \\ &= 2 \times \frac{1}{2} \times 12^2 \left[\frac{\pi}{2} - \sin 90^\circ \right] \simeq 82.19 \text{ cm}^2 \end{aligned}$$

**Example 5**

ABCD is a rectangle where $AB = 10$ cm. and $BC = 6$ cm. A circle of center A and radius length 10 cm. is drawn to cut $\overline{CD}$ at E. Find the surface area of the part bounded by $\overline{BC}$, $\overline{CE}$ and $\overline{BE}$.

Solution

In $\triangle ADE$:

$$\cos \angle EAD = \frac{6}{10}$$

$$\therefore m(\angle EAD) = \cos^{-1}\left(\frac{6}{10}\right) = 63^\circ 7' 48''$$

$$\therefore m(\angle EAB) = 90^\circ - 63^\circ 7' 48'' = 36^\circ 52' 12''$$

In the sector BAE:

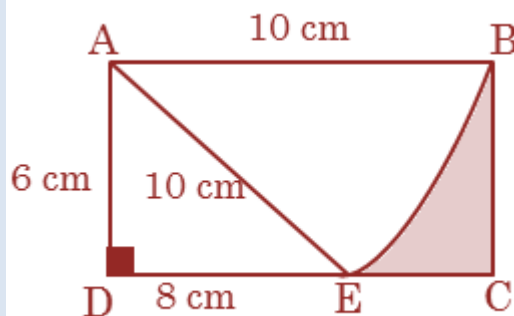
$$r = 10 \text{ cm.} \quad \theta^\circ = 36^\circ 52' 12'' \quad \theta^{\text{rad}} = 36^\circ 52' 12'' \times \pi \div 180^\circ = 0.644$$

$$\therefore \text{Area of sector BAE} = \frac{1}{2} r^2 \theta^{\text{rad}} = \frac{1}{2} \times 10^2 \times 0.644 \simeq 32 \text{ cm}^2$$

$$\therefore \text{Area of require part} = \text{Area of rectangle} - [\text{A. of sector BAE} + \text{A. of } \triangle ADE]$$

$$= L \times W - \left[\frac{1}{2} r^2 \theta^{\text{rad}} + \frac{1}{2} \times \text{base} \times \text{height} \right] \simeq 4 \text{ cm}^2$$

$$= 10 \times 6 - \left[32 + \frac{1}{2} \times 6 \times 8 \right] \simeq 4 \text{ cm}^2$$



**PRACTICE****Question 1**

$\overline{AB}$ is a chord of length 55 cm with a central angle of 95° . Find the area of the minor circular segment giving the answer to one decimal place.

Question 2

The area of a circle is 227 cm^2 and the central angle of a segment is 120° . Find the area of the segment giving the answer to two decimal places.

Question 3

What is the definition of a circular segment?

- ☐ A a region of a circle bounded by a chord and a central angle subtended by that arc
- ☐ B a region of a circle bounded between two chords and two arcs
- ☐ C an arc which is half of the circumference
- ☐ D a region of a circle bounded by two radii and an arc.
- ☐ E a region of a circle bounded by an arc and a chord passing through the end points of the arc.

**Question 4**

Which formula can be used to find the area of a circular segment, given a radius r and a central angle θ ?

- ☐ A $2r^2(\theta \text{ rad} - \sin \theta)$
- ☐ B $2r(\theta \text{ rad} + \sin \theta)$
- ☐ C $\frac{1}{2}r^2(\theta \text{ rad} + \sin \theta)$
- ☐ D $\frac{1}{2}r(\theta \text{ rad} - \sin \theta)$
- ☐ E $\frac{1}{2}r^2(\theta \text{ rad} - \sin \theta)$

Question 5

What is the total area of the minor and major circular segments of the same circle?

- ☐ A $\frac{1}{2}r^2(\theta - \sin \theta)$
- ☐ B $2\pi r$
- ☐ C $r^2(\theta - \sin \theta)$
- ☐ D πr^2

Question 6

A circle has a radius of 10 cm. A chord of length 14 cm is drawn. Find the area of the major segment, giving the answer to the nearest square centimeter.

- ☐ A 1 146 cm²
- ☐ B 187 cm²
- ☐ C 573 cm²
- ☐ D 287 cm²

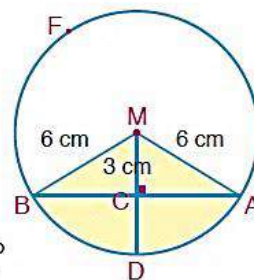


Home Work

1 In the figure drawn:

M is a circle of radius 6 cm, $\overline{MC} \perp \overline{AB}$, $MC = 3$ cm complete:

- A** The height of the circular segment ADB = cm
- B** The height of the major circular segment AFB = cm
- C** Measure of the angle of the minor circular segment ADB = $^{\circ}$
- D** Measure of the angle of the major circular segment AFB = $^{\circ}$
- E** Area of the triangle MAB = cm^2 .
- F** Area of the circular sector MADB in terms of π equals cm^2 .
- G** Area of the minor circular segment in terms of π equals cm^2 .

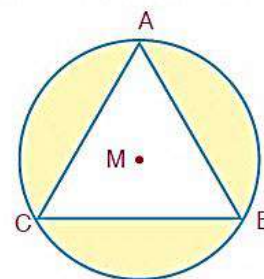


2 Find the area of the circular segment in which,

- A** Length of the radius of its circle is 12 cm and measure of its angle equals 1.4^{rad}
- B** The length of the radius of its circle equals 8 cm, and measure of its angle equals 135° .
.....
- C** The length of the radius of its circle equals 14 cm and the length of its arc equals 22 cm.
.....

3 In the figure drawn:

ABC is an equilateral triangle drawn in the circle M in which, the length of its radius equals 8 cm. find the area of each shaded circular segments.



4 Find the area of the major circular segment in which the length of its chord equals the length of the radius of its circle equals 12 cm.
.....

5 Find the area of the circular segment in which:

- A** The length of its chord equals 6 cm, and the length of the radius of its circle equals 5 cm.
- B** Its height equals 5 cm, and the length of the radius of its circle equals 10 cm.

6 A chord of length 8cm in a circle is at a distance 3cm from its centre. Find the area of the minor circular segment resulting from the intersection of this chord with the surface of the circle.
.....

**Lesson (5–7)****Areas****The Area of a triangle**

Area of the triangle = $\frac{1}{2}$ length of the base $\times$ height

The Area of a triangle in terms of the lengths of two sides and the included angle

In general:

Area of the triangle = half the product of the lengths of two sides $\times$ sine the included angle between them.

Example ①

Find the area of the ΔABC in which $AB = 9$ cm , $AC = 12$ cm , $m(\angle A) = 48^\circ$ approximating the result to the nearest hundredth.

Solution

$\therefore$ Area of the triangle $ABC = \frac{1}{2}$ length of the base $\times$ height

$\therefore$ Area of the triangle $ABC = \frac{1}{2} \times AB \times AC \times \sin \angle A$

$$= \frac{1}{2} \times 9 \times 12 \times \sin 48^\circ \simeq 40.13 \text{ cm}^2$$

Exercise ①:

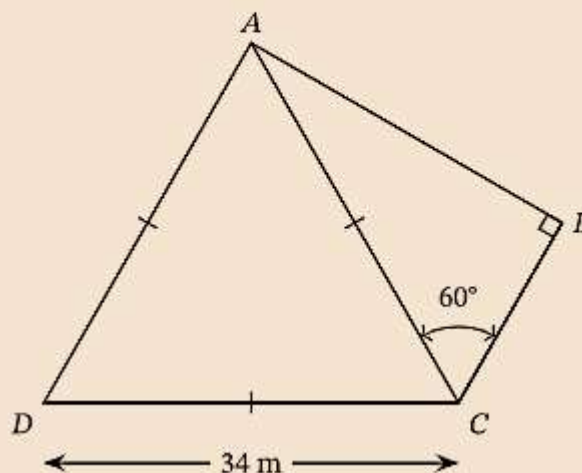
Find the area of the ΔABC in which $BC = 16$ cm , $BA = 22$ cm , $m(\angle B) = 63^\circ$ approximating the result to the nearest thousandth.

**PRACTICE (1)****Question 1**

ABC is a triangle, where $BC = 15$ cm, $AC = 25$ cm, and $m\angle C = 41^\circ$. Find the area of ABC , giving your answer to three decimal places.

Question 2

Find the area of the figure below giving the answer to three decimal places.

**Question 3**

ABC is a triangle where $AB = 5$ cm, $AC = 11$ cm, and $m\angle A = 107^\circ$. Find the area of the triangle giving the answer to two decimal places.

Question 4

ABC is an isosceles triangle where $AB = AC = 11$ cm and $m\angle A$ is 158° . Find the area of ABC giving the answer correct to two decimal places.



The Area of a Convex Quadrilateral

In general : Area of the quadrilateral in terms of the lengths of its diagonals and the included angle between them is:

Area of the quadrilateral = $\frac{1}{2}$ product of the lengths of its diagonals $\times$ sine the included angle between them

Example 2

Find the area of the quadrilateral in which the lengths of its diagonals are 12 cm, 16 cm and the measure of the included angle between them is 68° approximating the result to the nearest square centimeter.

Solution

$\therefore$ Area of the quadrilateral = $\frac{1}{2}$ product of diagonals $\times \sin$ (the included angle)

$\therefore$ Area of the quadrilateral = $\frac{1}{2} \times 12 \times 16 \times \sin 68^\circ \simeq 89 \text{ cm}^2$

Exercise ②:

Find the area of the quadrilateral in which the lengths of its diagonals are 32 cm, 46 cm and the measure of the included angle between them is 12° approximating the result to the nearest tenth.



The area of a regular polygon

Area of the polygon whose number of sides is n , and x is the length of its side $= \frac{1}{4} n x^2 \times \cot \frac{\pi}{n}$

Example ③

Find the area of the regular octagon in which the length of its side equals 6 cm approximating the result to the nearest hundredth.

Solution

$$\begin{aligned} \text{Area of the regular polygon} &= \frac{1}{4} n x^2 \times \cot \frac{\pi}{n} \\ &= \frac{1}{4} \times 8 \times 6^2 \times \cot \frac{\pi}{8} \\ &= 72 \times \frac{1}{\tan 22.5^\circ} \approx 173.8 \text{ cm}^2 \end{aligned}$$

Exercise ③:

Find the area of the regular pentagon in which the length of its side is 16 cm approximating the results of the nearest thousandth.

Name of polygon	Number of sides
Quadrilateral	4
Pentagon	5
Hexagon	6
Heptagon	7
Octagon	8
Nonagon	9
Decagon	10

Special name of regular polygons:

Triangle	Quadrilateral
Equilateral Triangle	Square

**PRACTICE (2)****Question 1**

Find the area of a regular 14-sided polygon given the side length is 21 cm. Give the answer to two decimal places.

Question 2

The perimeter of a regular pentagon is 85 cm. Find the area giving the answer to the nearest square centimetre.

- ☐ A 497 cm²
- ☐ B 262 cm²
- ☐ C 751 cm²
- ☐ D 1 395 cm²
- ☐ E 994 cm²

Question 3

Find the area of a regular hexagon with a side length of 35 cm giving the answer to two decimal places.

Question 4

The side length of a regular pentagon is 13 cm. Find the area of the pentagon giving the answer to three decimal places.

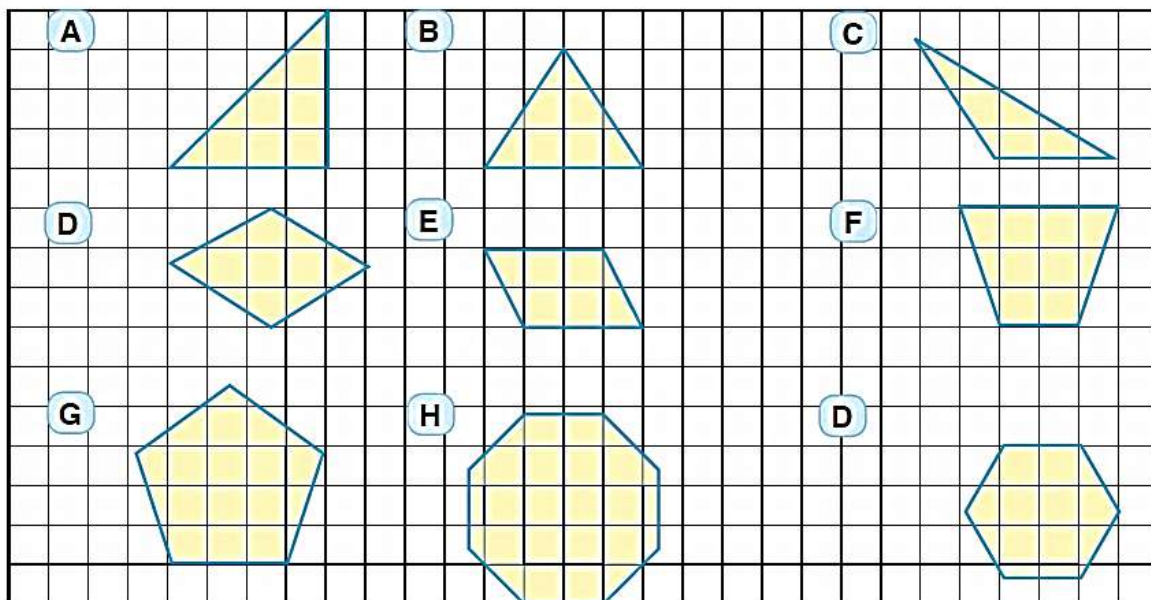
Question 5

The side of a regular octagon is 5 cm long. Find the area of the octagon giving the answer to two decimal places.



Home Work

- ① Find the area of each of the following figures, given that  expresses a unit of the area.



- ② Find the area of the triangle ABC in each of the following cases:

A $AB = 6\text{cm}$, $BC = 8\text{cm}$, $m(\angle B) = 90^\circ$

B $AC = 12\text{cm}$, length of perpendicular drawn from B to $\overline{AC}$ equals 7 cm.
.....

C $AB = 16\text{cm}$, $BC = 20\text{cm}$, $m(\angle B) = 46^\circ$

D $AB = 8\text{cm}$, $BC = 7\text{cm}$, $AC = 11\text{cm}$



③ Find the area of the figure ABCD in each of the following cases:

A Parallelogram in which $AB = 8\text{cm}$, $BC = 11\text{cm}$, $m(\angle B) = 60^\circ$

.....
.....

B Trapezium in which, length of its parallel bases $\overline{AD}$ and $\overline{BC}$ are equals to 7cm , 11cm respectively, the length of the perpendicular drawn from D to $\overline{BC}$ equals 6cm.

.....
.....

C A rhombus in which $AB = 8\text{cm}$, and measure of the included angle between two adjacent sides in it equals 58° .

.....
.....

④ Find the area of each of the following regular polygons approximating the result to the nearest tenth

A A regular pentagon of side length equals 16cm.

.....

B A regular hexagon of side length equals 12cm.

.....

Geometry

2023

2024

First Secondary

Second Term

Student Name:



Unit 3

**Lesson (3-1)****Scalars, Vectors and Directed Line Segment****Lesson objectives****Related Links**

-  Classifying and discriminating scalar and vector quantities.
-  Identify Concept of directed line segment, its direction and its norm.
-  Recognize the equivalent directed line segments.
-  Constructed of a directed line segment equivalent to another one in a coordinate plane.



The quantities which we deal with in our life are divided into two kinds of quantities:

① **Scalar quantities:**

Determined if we know their **magnitudes only**.

AS: length, time, mass, temperature, electric current tension and volume.

② **Vector quantities:**

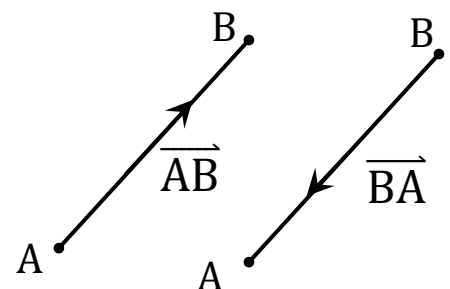
Determined if we know their **magnitudes** and **direction** of each of them.

AS: velocity, displacement, force and acceleration.

Directed line segment

The directed line segment is completely determined by determining 3 elements:

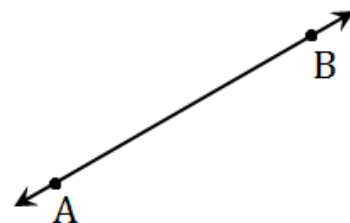
- ① The initial point.
- ② The terminal point.
- ③ The direction from the initial point to the terminal point.





Remarks

① $\overline{AB} = \overline{BA}$ while $\overrightarrow{AB} \neq \overrightarrow{BA}$ because they are different in the initial points and the terminal points.

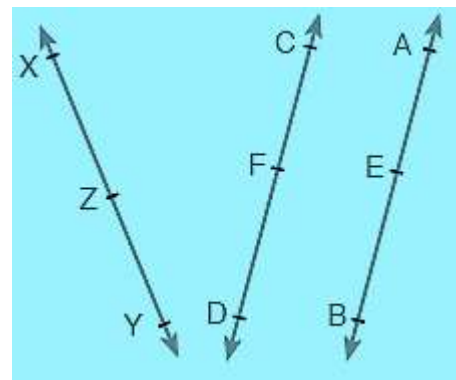


② We can consider the directed line segment as a displacement of a point moving on the straight line $\overleftrightarrow{AB}$ from the position A to the position B.

Example ①

In the figure opposite:

$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and each is not parallel to $\overleftrightarrow{XY}$,
 $E \in \overleftrightarrow{AB}$, $F \in \overleftrightarrow{CD}$, $Z \in \overleftrightarrow{XY}$.



Show whether the two rays have the same, opposite or different direction in each of the following:

① $\overrightarrow{AB}$, $\overrightarrow{DF}$

② $\overrightarrow{AB}$, $\overrightarrow{XY}$

③ $\overrightarrow{CD}$, $\overrightarrow{EB}$

④ $\overrightarrow{ZY}$, $\overrightarrow{ZX}$

⑤ $\overrightarrow{CF}$, $\overrightarrow{ZX}$

⑥ $\overrightarrow{ZX}$, $\overrightarrow{ZY}$

Solution

① $\overrightarrow{AB}$, $\overrightarrow{DF}$ have the opposite direction.

② $\overrightarrow{AB}$, $\overrightarrow{XY}$ have different direction.

③ $\overrightarrow{CD}$, $\overrightarrow{EB}$ have the same direction.

④ $\overrightarrow{ZY}$, $\overrightarrow{ZX}$ have opposite direction.

⑤ $\overrightarrow{CF}$, $\overrightarrow{ZX}$ have different direction.

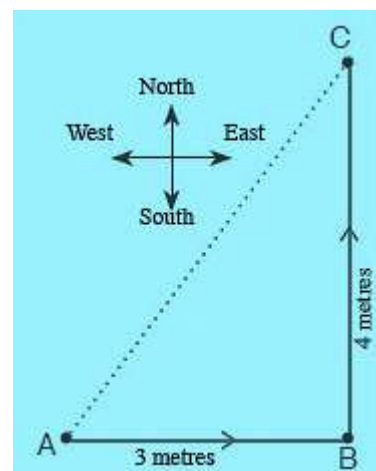
⑥ $\overrightarrow{ZX}$, $\overrightarrow{ZY}$ have opposite direction.

**Notice that:**

◆ **Distance** is a scalar quantity which is the result of $AB + BC$ or $CB + BA$.

◆ **Displacement** is the distance between the starting and ending points only and in direction from A to C.

i.e. to describe the displacement, its magnitude AC and its direction from A to C must be determined.



For example:

In the figure opposite:

Calculate the distance and displacement covered when a body moves from the point A to the point C, then returns to point B.



■ **Distance** = $AC + CB = 10 + 4 = 14$ cm.

■ **Displacement** = $AC - CB = 10 - 4 = 6$ cm. (= AB)

Definition of equivalence of two directed line segments

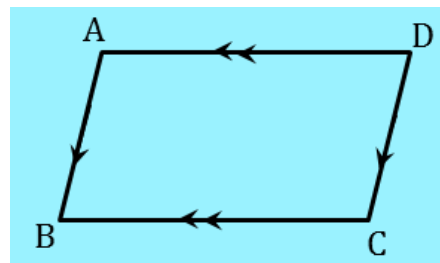
It is said the two directed line segments are equivalent if they have:

① The same length.

② The same direction.

Remark

If ABCD is a parallelogram,
then $\overrightarrow{AB}$ is equivalent to $\overrightarrow{DC}$
and $\overrightarrow{DA}$ is equivalent to $\overrightarrow{CB}$



**Example 2**

ABCD is a parallelogram; its diagonals are intersecting at M.

First: Determine the directed line segments (if existed) which are equivalent to:

- ① $\overrightarrow{AB}$ ② $\overrightarrow{CD}$ ③ $\overrightarrow{BC}$ ④ $\overrightarrow{AM}$ ⑤ $\overrightarrow{MD}$

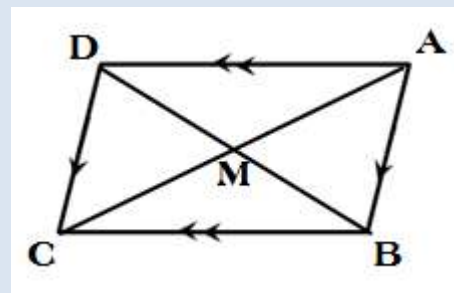
Second: Show why the following directed line segments are not equivalent?

- ① $\overrightarrow{AM}$, $\overrightarrow{AC}$ ② $\overrightarrow{BA}$, $\overrightarrow{DC}$ ③ $\overrightarrow{BM}$, $\overrightarrow{DM}$

Solution

First:

- | | | |
|-------------------------|----------------|-----------------------|
| ① $\overrightarrow{AB}$ | equivalent to: | $\overrightarrow{DC}$ |
| ② $\overrightarrow{CD}$ | equivalent to: | $\overrightarrow{BA}$ |
| ③ $\overrightarrow{BC}$ | equivalent to: | $\overrightarrow{AD}$ |
| ④ $\overrightarrow{AM}$ | equivalent to: | $\overrightarrow{MC}$ |
| ⑤ $\overrightarrow{MD}$ | equivalent to: | $\overrightarrow{BM}$ |



Second:

- | | | |
|---|--|----------------------------|
| ① $\overrightarrow{AM}$, $\overrightarrow{AC}$ | are not equivalent because: $AM \neq AC$ | (have different length) |
| ② $\overrightarrow{BA}$, $\overrightarrow{DC}$ | are not equivalent because: $\overrightarrow{BA} \neq \overrightarrow{DC}$ | (have different direction) |
| ③ $\overrightarrow{BM}$, $\overrightarrow{DM}$ | are not equivalent because: $\overrightarrow{BM} \neq \overrightarrow{DM}$ | (have different direction) |

◆ **The norm of the directed line segment:** norm of $\overrightarrow{AB}$ is the length of $\overrightarrow{AB}$ and is denoted by the symbol $\|\overrightarrow{AB}\|$.

i.e. $\|\overrightarrow{AB}\| = \|\overrightarrow{BA}\| = AB$



Example 3

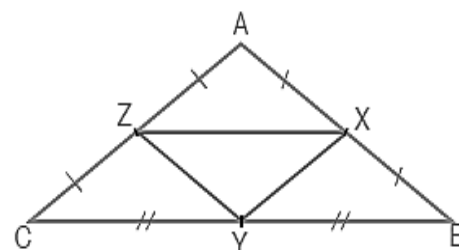
In the opposite figure:

ABC is a triangle in which $AB = AC$

X, Y and Z are midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ respectively

First: Which of the following statements are true?

- ① $\|\overrightarrow{XY}\| = \|\overrightarrow{YZ}\|$
- ② $\overrightarrow{XY}$ is equivalent to $\overrightarrow{YZ}$
- ③ $\overrightarrow{BY}$ is equivalent to $\overrightarrow{ZX}$



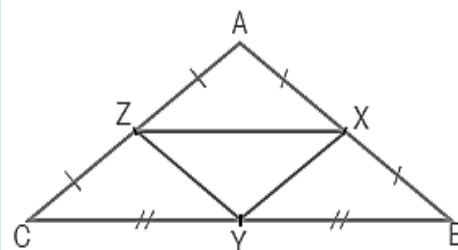
Second: Write all directed line segments (if found) which are equivalent to:

- ① $\overrightarrow{BX}$
- ② $\overrightarrow{AZ}$
- ③ $\overrightarrow{XZ}$
- ④ $\overrightarrow{CY}$
- ⑤ $\overrightarrow{XY}$
- ⑥ $\overrightarrow{ZY}$

Solution

First: Which of the following statements are true?

- ① $\|\overrightarrow{XY}\| = \|\overrightarrow{YZ}\|$ [True]
- ② $\overrightarrow{XY}$ is equivalent to $\overrightarrow{YZ}$ [False]
- ③ $\overrightarrow{BY}$ is equivalent to $\overrightarrow{ZX}$ [False]



Second: Write all directed line segments (if found) which are equivalent to:

- ① $\overrightarrow{BX}$
- ② $\overrightarrow{AZ}$
- ③ $\overrightarrow{XZ}$
- ④ $\overrightarrow{CY}$
- ⑤ $\overrightarrow{XY}$
- ⑥ $\overrightarrow{ZY}$

- $\overrightarrow{XA}$
- $\overrightarrow{ZC}$
- $\overrightarrow{BY}$
- $\overrightarrow{YB}$
- $\overrightarrow{AZ}$
- $\overrightarrow{AX}$

- $\overrightarrow{YZ}$
- $\overrightarrow{XY}$
- $\overrightarrow{YC}$
- $\overrightarrow{ZX}$
- $\overrightarrow{ZC}$
- $\overrightarrow{XB}$



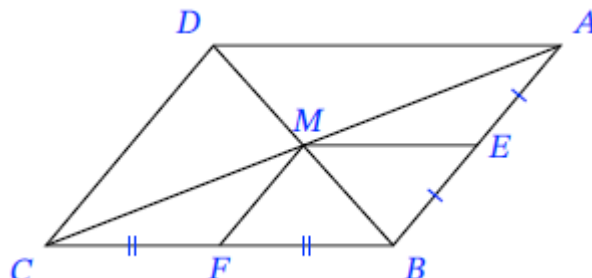
PRACTICE

Choose the correct answer

Q (1):

In the given parallelogram ABCD, $\overline{AC} \cap \overline{BD} = \{M\}$, E is the midpoint of $\overline{AB}$, and F is the midpoint of $\overline{BC}$, then $\overline{AM}$ is equivalent to

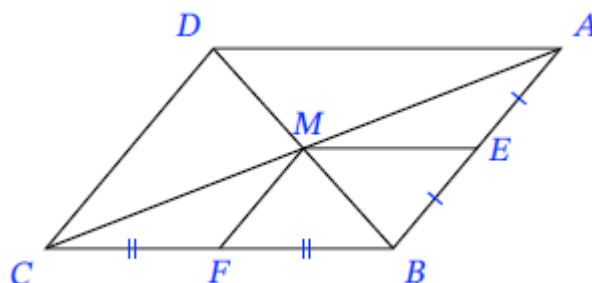
- (a) $\overline{AC}$
- (b) $\overline{MA}$
- (c) $\overline{CM}$
- (d) $\frac{1}{2}\overline{AC}$



Q (2):

In the given parallelogram ABCD, $\overline{AC} \cap \overline{BD} = \{M\}$, E is the midpoint of $\overline{AB}$, and F is the midpoint of $\overline{BC}$, then $\overline{ME}$ is equivalent to

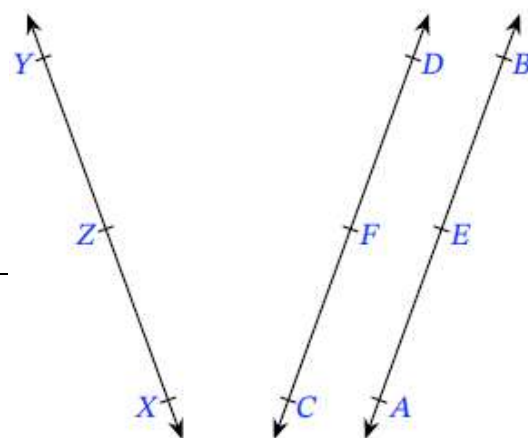
- (a) $\overline{BF}$
- (b) $\frac{1}{2}\overline{BC}$
- (c) $\overline{FC}$
- (d) $\frac{1}{2}\overline{DA}$



Q (3):

In the given figure: $\overline{AB} \parallel \overline{CD}$, then which statement is true for: $\overline{AE}$ & $\overline{CD}$

- (a) opposite
- (b) same
- (c) different
- (d) equivalent



Q (4):

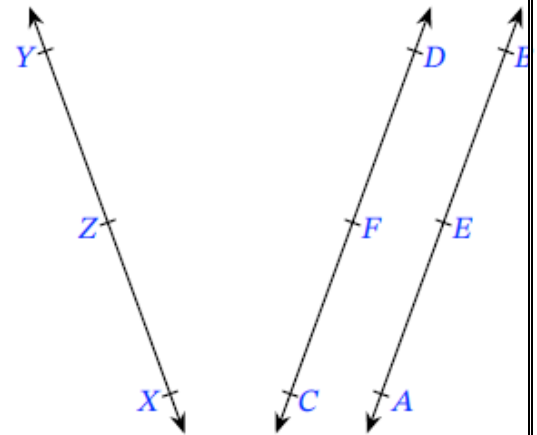
In the given figure: $\overline{AB} \parallel \overline{CD}$, then which statement is true for: $\overline{AE}$ & $\overline{DF}$

- (a) opposite
- (b) same
- (c) different
- (d) equivalent



Q (5): In the given figure: $\overrightarrow{AB} \parallel \overrightarrow{CD}$, then which statement is true for: $\overrightarrow{AB}$ & $\overrightarrow{XY}$

- (a) opposite
- (b) same
- (c) different
- (d) equivalent



Q (6): In the given figure: $\overrightarrow{AB} \parallel \overrightarrow{CD}$, then which statement is true for: $\overrightarrow{CF}$ & $\overrightarrow{DF}$

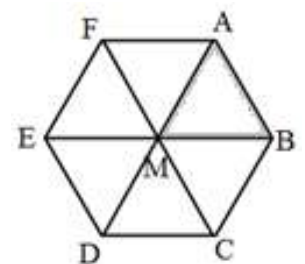
- (a) opposite
- (b) same
- (c) different
- (d) equivalent

Q (7): Select all the statements that must be true if $\vec{u}$ and $\vec{v}$ are equivalent vectors

- (a) The initial point of u is the terminal point of v.
- (b) u and v have the same terminal point.
- (c) u and v have the same initial point.
- (d) u and v have the same direction and $\|\vec{u}\| = \|\vec{v}\|$

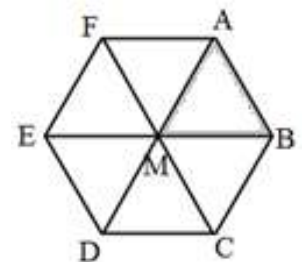
Q (8): In the given figure: ABCDEF is a regular hexagon then which is equivalent to $\overrightarrow{AF}$

- (a) $\overrightarrow{ME}$
- (b) $-\overrightarrow{DC}$
- (c) $\frac{1}{2}\overrightarrow{BE}$
- (d) All the previous



Q (9): In the given figure: ABCDEF is a regular hexagon then which is not equivalent to $\overrightarrow{DM}$

- (a) $\overrightarrow{EF}$
- (b) $-\overrightarrow{MA}$
- (c) $\frac{1}{2}\overrightarrow{DA}$
- (d) $-\overrightarrow{CB}$





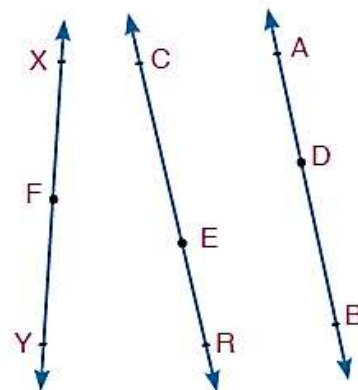
Home Work

① Complete the following statements to be true:

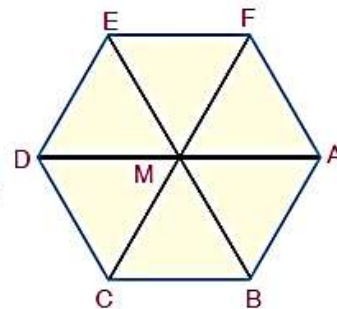
- (A) To defined Scalar quantity completely you should know
- (B) To defined Vector quantity completely you should know
- (C) The directed line segment is a line segment which has,,
- (D) Two directed line segments are equivalent if they have.....

② In the figure opposite: $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CE}$ and each of them is not parallel to $\overleftrightarrow{XY}$. join each of the following statements with a suitable one .

- (A) $\overrightarrow{AD}$, $\overrightarrow{AB}$ 1- have the same direction (B) $\overrightarrow{FX}$, $\overrightarrow{XY}$
- (C) $\overrightarrow{DA}$, $\overrightarrow{ER}$ 2- have different direction (D) $\overrightarrow{CE}$, $\overrightarrow{AB}$
- (E) $\overrightarrow{BD}$, $\overrightarrow{YF}$ 3- have opposite direction (F) $\overrightarrow{CR}$, $\overrightarrow{XY}$



③ In the figure opposite ABCDEF, is a regular hexagon, its centre is M, Complete the following:



- (A) $\overrightarrow{AB}$ is equivalent to and equivalent to and equivalent to
- (B) $\overrightarrow{MD}$ is equivalent to and equivalent to and equivalent to
- (C) $\overrightarrow{CD}$ is equivalent to and equivalent to and equivalent to
- ④ ABCD is a square , its diagonals are intersecting at M, write all directed equivalent line segments determined in the figure.
-
- ⑤ In a coordinate orthogonal plane : If A(4, -3) , B (4, 4), C (-3, -1), $\overrightarrow{BA}$, $\overrightarrow{CD}$, $\overrightarrow{OM}$, $\overrightarrow{NO}$ are equivalent directed line segments , where O is the origin. Find the coordinates of each of D, M , and N.



- ⑥ On the lattice : A(3, -2), B (6, 2) , C (1, 3) , D (4, 7) :
- Ⓐ Find $\|\vec{AB}\|$, $\|\vec{CD}\|$
 - Ⓑ Prove that: $\vec{AB}$ is equivalent to $\vec{CD}$
 - Ⓒ If each of the directed line segments $\vec{BC}$, $\vec{AM}$, $\vec{ND}$, $\vec{OR}$ are equivalent, Find the coordinates of each of M, N and R where O is the origin.
- ⑦ On the lattice: A(2, 3) , B (-3, 1), C (5, -1)
- Ⓐ Draw $\vec{CD}$, such that it is equivalent to $\vec{AB}$ and determine the coordinates of D.
 - Ⓑ Determine the coordinates of the point M which is the mid-point of $\vec{BC}$, then determine the directed line segments which are equivalent to each of :

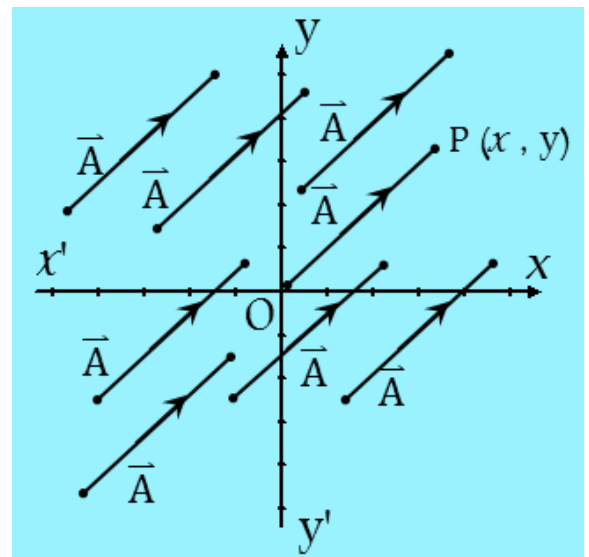
First: $\vec{BM}$ Second: $\vec{AM}$ Third: $\vec{AC}$ Fourth: $\vec{DB}$
 - Ⓒ Is the figure ACDB a parallelogram? Explain your answer.

**Lesson (3-2)****Vectors****Lesson objectives****Related Links**

- Find the position vector of a given point with respect to the origin in the orthogonal coordinate plane.
- Put a vector in the polar form.
- Find the norm of a vector.
- Identify Concept of equivalent two vectors and solving exercises on it.
- Express the vector in terms of the fundamental unit vector.
- Recognize Condition of two parallel vectors.

**Position Vector**

Each vector $\vec{A} = (x, y) \in \mathbb{R}^2$ can be represented geometrically by many directed line segments which are equivalent and each of them is equivalent to the position vector of the point $P(x, y)$.

**Definition of addition of two vectors**

If: $\vec{A} = (x_1, y_1)$, $\vec{B} = (x_2, y_2)$, then: $\vec{A} + \vec{B} = (x_1 + x_2, y_1 + y_2)$



Definition multiplying a vector by a real number

If: $\vec{A} = (x, y)$, $k \in \mathbb{R}$, then: $k \vec{A} = k(x, y) = (kx, ky)$

for example: $3(2, -5) = (6, -15)$, $\frac{1}{2}(4, 9) = (2, \frac{9}{2})$, $4(0, 0) = (0, 0)$, $-2(3, -4) = (-6, 8)$

Properties of multiplication operation on vectors:

Distributive property	First: for every $\vec{A}, \vec{B} \in \mathbb{R}^2$, $K \in \mathbb{R}$ then: $K(\vec{A} + \vec{B}) = K\vec{A} + K\vec{B}$ Second: for every $\vec{A} \in \mathbb{R}^2$, $K_1, K_2 \in \mathbb{R}$ then: $(K_1 + K_2)\vec{A} = K_1\vec{A} + K_2\vec{A}$
Association property	for every $\vec{A} \in \mathbb{R}^2$, $K_1, K_2 \in \mathbb{R}$ then: $(K_1 K_2)\vec{A} = K_1(K_2\vec{A})$
Elimination property	for every $\vec{A}, \vec{B} \in \mathbb{R}^2$, $K \in \mathbb{R}^*$ If $K\vec{A} = K\vec{B}$ then: $\vec{A} = \vec{B}$ and vice versa

Example 1

If: $\vec{A} = (3, -1)$, $\vec{B} = (2, 5)$ and $\vec{C} = (-4, 2)$, find each of the following vectors:

- ① $2\vec{A} - 3\vec{B}$ ② $\frac{1}{2}(4\vec{A} + 2\vec{B} - \vec{C})$ ③ $2\vec{O} - 3(\vec{C} + \vec{A})$

Solution

$$\textcircled{1} \quad 2\vec{A} - 3\vec{B} = 2(3, -1) - 3(2, 5) = (6, -2) - (6, 15) = \mathbf{(0, -17)}$$

$$\begin{aligned} \textcircled{2} \quad \frac{1}{2}(4\vec{A} + 2\vec{B} - \vec{C}) &= 2\vec{A} + \vec{B} - \frac{1}{2}\vec{C} = 2(3, -1) + (2, 5) - \frac{1}{2}(-4, 2) \\ &= (6, -2) + (2, 5) - (-2, 1) = \mathbf{(10, 2)} \end{aligned}$$

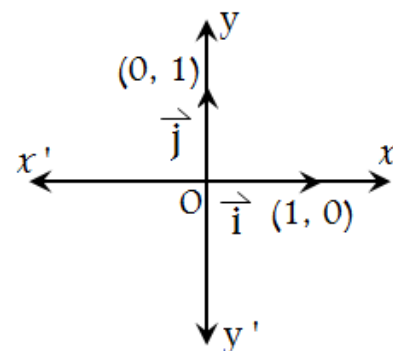
$$\begin{aligned} \textcircled{3} \quad 2\vec{O} - 3(\vec{C} + \vec{A}) &= 2(0, 0) - 3[(-4, 2) + (3, -1)] \\ &= (0, 0) - 3(-1, 1) = -3(-1, 1) = \mathbf{(3, -3)} \end{aligned}$$



Vectors and coordinates

The two fundamental unit vectors: $\vec{i}$ and $\vec{j}$

If we have a system of rectangular coordinates axes in the plane, O is the origin point, then:



① The vector $\vec{i} = (1, 0)$ which represents the position vector of the point (1, 0) is named the unit vector in the direction of $\overrightarrow{OX}$ (the positive direction of x-axis).

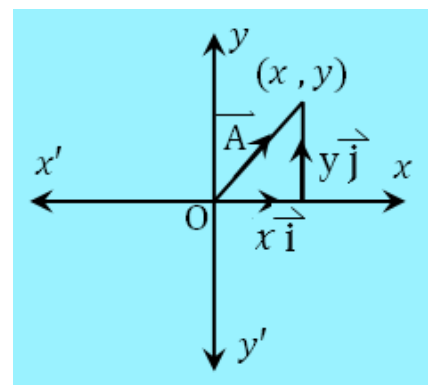
② The vector $\vec{j} = (0, 1)$ which represents the position vector of the point (0, 1) is named the unit vector in the direction of $\overrightarrow{OY}$ (the positive direction of y-axis).

Expressing the vector in terms of the two fundamental unit vectors:

If $\vec{A}$ is a vector in the plane where $\vec{A} = (x, y)$, then we can express the vector $\vec{A}$ in terms of $\vec{i}$ and $\vec{j}$ (the two unit vectors) as follows:

$$\vec{A} = (x, y) = (x, 0) + (0, y) = x(1, 0) + y(0, 1)$$

$$\therefore \vec{A} = x\vec{i} + y\vec{j}$$



Example 2

Express each of the following vectors in terms of the fundamental unit vectors:

① $\vec{M} = (-3, 4)$

② $\vec{N} = (5, -12)$

③ $\vec{L} = (-3, -6)$

④ $\vec{Z} = (-7, 0)$

Solution

① $\vec{M} = (-3, 4) = -3\vec{i} + 4\vec{j}$

② $\vec{N} = (5, -12) = 5\vec{i} - 12\vec{j}$

③ $\vec{L} = (-3, -6) = -3\vec{i} - 6\vec{j}$

④ $\vec{Z} = (-7, 0) = -7\vec{i}$

**Example ③**

Find in terms of the two fundamental unit vectors the vector which expresses each of the following:

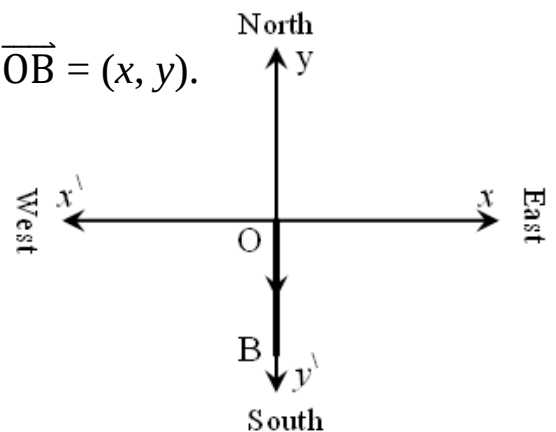
- ① The displacement of a body a distance 60 cm in the south direction.
- ② A force of magnitude 30 kg.wt acts on a particle in the direction 60° north of west.

Solution

- ① Let the position vector of the displacement be $\overrightarrow{OB} = (x, y)$.

$$\therefore x = 0, \quad y = -60$$

$$\therefore \vec{B} = -60 \vec{i}$$

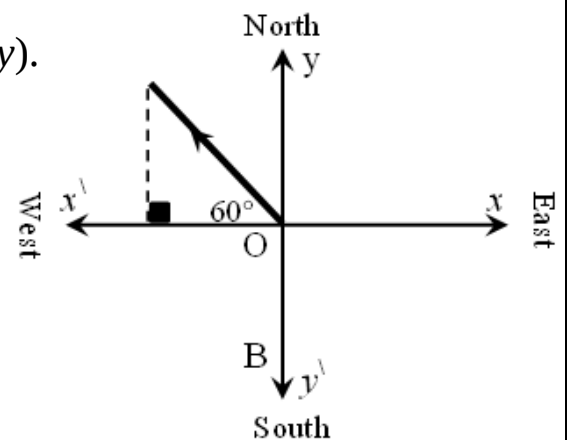


- ② Let the position vector of the force be $\overrightarrow{OC} = (x, y)$.

$$\therefore x = -30 \cos 60^\circ = -15$$

$$, y = 30 \sin 60^\circ = 15\sqrt{3}$$

$$\therefore \vec{C} = -15 \vec{i} + 15\sqrt{3} \vec{j}$$

**Definition of the norm of a vector (the length of the vector)**

If: $\vec{A} = (x, y)$, then the real number $\sqrt{x^2 + y^2}$ is called the norm (or the length) of the vector $\vec{A}$ **i.e.** $\|\vec{A}\| = \sqrt{x^2 + y^2}$

**Remark**

If: $\vec{A} = (x_1, y_1)$, $\vec{B} = (x_2, y_2)$, then: $\vec{AB} = \vec{B} - \vec{A} = (x_2 - x_1, y_2 - y_1)$

Example 4

- ❶ If: $\vec{A} = (4, 0)$, $\vec{B} = (0, 3)$, then find $\|\vec{AB}\|$
- ❷ If: $\vec{A} = 5\vec{i} + 4\vec{j}$, $\vec{B} = 5\vec{i} - 4\vec{j}$, then find $\vec{AB}$
- ❸ If: $\vec{AB} = (2, -3)$ and $\vec{A} = (3, 4)$ find vector $\|\vec{B}\|$
- ❹ If: $\|\vec{AB}\| = 4$ and $\vec{A} = (3, 4)$ and $\vec{B} = (k, 8)$ find the values of k

Solution

$$\text{❶} \because \vec{AB} = \vec{B} - \vec{A} = (0, 3) - (4, 0) = (-4, 3)$$

$$\therefore \|\vec{AB}\| = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + 3^2} = 5$$

$$\text{❷} \because \vec{AB} = \vec{B} - \vec{A} = (5, -4) - (5, 4) = (0, -8) = -8\vec{j}$$

$$\text{❸} \because \vec{AB} = \vec{B} - \vec{A}$$

$$\therefore \vec{B} = \vec{AB} + \vec{A}$$

$$\therefore \vec{B} = (2, -3) + (3, 4) = (5, 1)$$

$$\therefore \|\vec{B}\| = \sqrt{x^2 + y^2} = \sqrt{5^2 + 1^2} = \sqrt{26}$$

$$\text{❹} \because \vec{AB} = \vec{B} - \vec{A} = (k, 8) - (3, 4) = (k - 3, 4)$$

$$\therefore \|\vec{AB}\| = 4$$

$$\therefore \|\vec{AB}\| = \sqrt{x^2 + y^2} = 4$$

$$\therefore \sqrt{(k - 3)^2 + 4^2} = 4$$

$$\therefore (k - 3)^2 + 4^2 = 4^2$$

$$\therefore (k - 3)^2 = 0$$

$$\therefore k = 3$$



Parallel & Perpendicular two vectors

If: $\vec{A} = (x_1, y_1)$ and $\vec{B} = (x_2, y_2)$, then:

$$\vec{A} // \vec{B} \quad \text{if} \quad m_1 = m_2 \quad \vec{A} \perp \vec{B} \quad \text{if} \quad m_1 \times m_2 = -1$$

where $m = \frac{y}{x}$

Example 5

If $\vec{A} = (2, 4)$, $\vec{B} = (-6, 3)$, $\vec{C} = (4, 8)$. **Prove that:** $\vec{A} \perp \vec{B}$, $\vec{A} // \vec{C}$, $\vec{B} \perp \vec{C}$

Solution

$\vec{A} \perp \vec{B}$	Because:	$m_A = \frac{y}{x} = \frac{4}{2} = 2$	$m_B = \frac{y}{x} = \frac{3}{-6} = -\frac{1}{2}$
		$\therefore m_A \times m_B = -1$	$\therefore \vec{A} \perp \vec{B}$
$\vec{A} // \vec{C}$	Because:	$m_A = \frac{y}{x} = \frac{4}{2} = 2$	$m_C = \frac{y}{x} = \frac{8}{4} = 2$
		$\therefore m_A = m_C$	$\therefore \vec{A} // \vec{C}$
$\vec{B} \perp \vec{C}$	Because:	$m_B = \frac{y}{x} = \frac{3}{-6} = -\frac{1}{2}$	$m_C = \frac{y}{x} = \frac{8}{4} = 2$
		$\therefore m_B \times m_C = -1$	$\therefore \vec{B} \perp \vec{C}$

Example 6

If $\vec{A} = (2, 5)$, $\vec{B} = (k, -4)$, find the value of k when: ① $\vec{A} // \vec{B}$ ② $\vec{A} \perp \vec{B}$

Solution

① $\therefore \vec{A} // \vec{B}$	$\therefore m_A = m_B$	$\therefore \frac{5}{2} = \frac{-4}{k}$	$\therefore k = \frac{-8}{5}$
② $\vec{A} \perp \vec{B}$	$\therefore m_A \times m_B = -1$	$\therefore \frac{5}{2} \times \frac{-4}{k} = -1$	
	$\therefore \frac{-4}{k} = -1 \div \frac{5}{2}$	$\frac{-4}{k} = \frac{-2}{5}$	$\therefore k = 10$



Notice that

If $\vec{M} = (x, y)$, $K \in \mathbb{R}$

then: $K \vec{M} = K(x, y) = (Kx, Ky)$

If $\vec{M}$ is a non-zero vector, $K \neq 0$ then: $\vec{M} \parallel K \vec{M}$

and: $\|K \vec{M}\| = |K| \cdot \|\vec{M}\|$

where the direction of $K \vec{M}$ is the same as the direction of $\vec{M}$ for every $K > 0$
the direction of $K \vec{M}$ is opposite to the direction of $\vec{M}$ for every $K < 0$

Example 7

Find the value of k in each of the following:

① $k \|\vec{A}\| = \|5 \vec{A}\|$

② $2\|k \vec{A}\| = \|-7 \vec{A}\|$

③ $\|3k \vec{A}\| = \|6 \vec{A}\|$

Solution

① $k \|\vec{A}\| = \|5 \vec{A}\|$

$\therefore k = 5$

② $2\|k \vec{A}\| = \|-7 \vec{A}\|$

$\therefore \pm 2k = 7$

$\therefore k = \pm \frac{7}{2}$

③ $\|3k \vec{A}\| = \|6 \vec{A}\|$

$\therefore \pm 3k = 6$

$\therefore k = \pm \frac{6}{3} = \pm 2$

Example 8

If: $\vec{A} = (6\sqrt{3}, 60^\circ)$ Find: $\vec{A}$ in fundamental unit form

Solution

$x = 6\sqrt{3} \cos 60^\circ = 3\sqrt{3}$

$y = 6\sqrt{3} \sin 60^\circ = 9$

$\vec{A} = 3\sqrt{3} \vec{i} - 9 \vec{j}$



Polar form

If: $\vec{A} = (x, y)$, then $\vec{A} = (\|\vec{A}\|, \theta^{rad})$ where $\tan \theta = \frac{y}{x}$

Example 9

If $\vec{A} = (6, -2)$, $\vec{B} = (4, 3)$. Express $\vec{C} = (11, 5)$ in terms of $\vec{A}$, and $\vec{B}$

Solution

Let $\vec{C} = k \vec{A} + m \vec{B}$

$$\therefore (11, 5) = k(6, -2) + m(4, 3)$$

$$\therefore (11, 5) = (6k, -2k) + (4m, 3m)$$

$$\therefore 6k + 4m = 11$$

→ ①

$$\therefore -2k + 3m = 5$$

→ ②

By solving together

MODE

EQN

1

$$\therefore k = \frac{1}{2} \quad \& \quad m = 2$$

$$\therefore \vec{C} = \frac{1}{2} \vec{A} + 2 \vec{B}$$

**PRACTICE****Choose the correct answer**

Q (1): What is the magnitude of the vector $\overline{AB}$, where $A = (5, -9)$ and $B = (9, 1)$?

- Ⓐ $2\sqrt{29}$
- Ⓑ 116
- Ⓒ $\sqrt{14}$
- Ⓓ $\sqrt{10}$

Q (2): A body moved 190 cm due east, where i and j are two unit vectors in the east and north directions, respectively. Express its displacement

- Ⓐ $-190i$ cm
- Ⓑ $190j$ cm
- Ⓒ $-190j$ cm
- Ⓓ $190i$ cm

Q (3): What is the name given to the vector $(0, 0)$?

- Ⓐ the empty vector
- Ⓑ the unit vector
- Ⓒ the zero vector
- Ⓓ the origin vector

Q (4): Given that $A = (0, 1)$ and $B = (-3, -6)$, find $\frac{3}{2}(A - B)$.

- Ⓐ $\left(-\frac{9}{2}, -\frac{15}{2}\right)$
- Ⓑ $\left(\frac{9}{2}, \frac{21}{2}\right)$
- Ⓒ $\left(-\frac{9}{2}, \frac{21}{2}\right)$
- Ⓓ $\left(\frac{9}{2}, -\frac{15}{2}\right)$

Q (5): Given that $A = (8, 3)$ and $B = (-5, 3)$, find $\frac{1}{2}(A - B)$.

- Ⓐ $\left(\frac{3}{2}, 3\right)$
- Ⓑ $\left(\frac{13}{2}, 0\right)$
- Ⓒ $\left(\frac{3}{2}, 0\right)$
- Ⓓ $\left(\frac{13}{2}, 3\right)$



Q (6): Given that $A = (0, -1)$ and $\|k \vec{A}\| = 12$, then $k = \dots\dots\dots$

- Ⓐ $\frac{1}{12}, -\frac{1}{12}$
- Ⓑ $\frac{1}{12}$
- Ⓒ 12
- Ⓓ $12, -12$

Q (7): Given that $A = (-4, -1)$ and $B = (-2, -1)$, express $C = (-8, -1)$ in terms of A & B .

- Ⓐ $5A - 6B$
- Ⓑ $-A + 6B$
- Ⓒ $7A + 10B$
- Ⓓ $3A - 2B$

Q (8): If $\vec{A} = (7, 60^\circ)$ is the position vector, in polar form, find the xy -coordinates of $\vec{A}$.

- Ⓐ $\left(\frac{7\sqrt{3}}{2}, \frac{7}{2}\right)$
- Ⓑ $\left(\frac{7\sqrt{3}}{2}, \frac{7\sqrt{3}}{2}\right)$
- Ⓒ $\left(\frac{7}{2}, \frac{7}{2}\right)$
- Ⓓ $\left(\frac{7}{2}, \frac{7\sqrt{3}}{2}\right)$

Q (9): If $\vec{C} = (4\sqrt{3}, \frac{3\pi}{4})$ is the position vector, in polar form, find the xy -coordinates of $\vec{C}$.

- Ⓐ $(2\sqrt{6}, -2\sqrt{6})$
- Ⓑ $(-4\sqrt{6}, 2\sqrt{6})$
- Ⓒ $(-2\sqrt{6}, 4\sqrt{6})$
- Ⓓ $(-2\sqrt{6}, 2\sqrt{6})$



Q (10): Given the point A $(-4\sqrt{3}, 4)$, express, in polar form.

- Ⓐ $(8, \frac{11\pi}{6})$
- Ⓑ $(8, \frac{11\pi}{3})$
- Ⓒ $(8\sqrt{2}, \frac{11\pi}{12})$
- Ⓓ $(8, \frac{11\pi}{12})$

Q (11): Given the point A $(3\sqrt{3}, -9)$, express, in polar form.

- Ⓐ $(6\sqrt{3}, \frac{5\pi}{3})$
- Ⓑ $(12, \frac{5\pi}{6})$
- Ⓒ $(6\sqrt{3}, \frac{5\pi}{6})$
- Ⓓ $(12, \frac{5\pi}{3})$

Q (12): Given that A = $(-6, -15)$, B = $(k, -10)$, and $\vec{A} \parallel \vec{B}$, find the value of k

- Ⓐ 4
- Ⓑ 9
- Ⓒ -4
- Ⓓ -9

Q (13): Trapezium ABCD has vertices A $(10, 11)$, B $(k, 8)$, C $(4, -12)$ and D $(-2, 6)$.
Given that $\vec{AB} \parallel \vec{CD}$, find the value of k.

- Ⓐ 11
- Ⓑ 46
- Ⓒ -26
- Ⓓ 9

Home Work

- 1 On the lattice : $A(3, -4)$, $B(-12, 5)$, $C(-3, -6)$, Find the position vector for each of the points A , B , C with respect to the origin $O(0, 0)$, then find the norm of each of them.
- 2 Express each of the following vectors in terms of the fundamental unit vectors. Then find the norm of each of them:
 - A $\vec{M} = (-4, -3)$
 - B $\vec{N} = (8, -6)$
 - C $\vec{F} = (-5, -12)$
 - D $\vec{A} = (0, 2\sqrt{2})$
 - E $\vec{B} = (-3\sqrt{3}, 0)$
 - F $\vec{C} = (\sqrt{2}, -3\sqrt{2})$
- 3 Find the polar form of each of the following vectors:
 - A $\vec{M} = 8\sqrt{3}\vec{i} + 8\vec{j}$
 - B $\vec{N} = 3\sqrt{2}\vec{i} + 3\sqrt{2}\vec{j}$
- 4 If $\vec{A} = (3, -2)$, $\vec{B} = (-2, 5)$, $\vec{C} = (0, 11)$:
 - A Write each of the following vectors in terms of the fundamental unit vectors $2\vec{B}$, $3\vec{C}$, $\vec{A} + \vec{B} - \vec{C}$, $\frac{1}{2}(\vec{B} + \vec{C})$
 - B Express $\vec{C}$ in terms of $\vec{A}$ and $\vec{B}$
- 5 Find in terms of the fundamental unit vectors, the vector which expresses:
 - A A uniform speed of magnitude 60 km/h in west direction.
 - B A force of magnitude 20 kgm wt . acts on a body in the direction 30° south of east
 - C A displacement of a body a distance 40cm in the direction north west .



⑥ If $\vec{M} = (5, 1)$, $\vec{N} = (4, -20)$, $\vec{L} = (-10, -2)$ Prove that:

Ⓐ $\vec{M} \perp \vec{N}$

Ⓑ $\vec{M} \parallel \vec{L}$

Ⓒ $\vec{N} \perp \vec{L}$

⑦ If $\vec{M} = 3\vec{i} + 4\vec{j}$, $\vec{N} = -6\vec{i} - 8\vec{j}$

$\vec{L} = a\vec{i} - 8\vec{j}$, $\vec{F} = 4\vec{i} + b\vec{j}$

Ⓐ Prove that $\vec{M} \parallel \vec{N}$

.....

Ⓑ Find $a \in \mathbb{R}$, if $\vec{M} \parallel \vec{L}$

.....

Ⓒ Find $b \in \mathbb{R}$, if $\vec{F} \perp \vec{N}$

.....

Ⓓ Is $\vec{F} \perp \vec{M}$? explain your answer

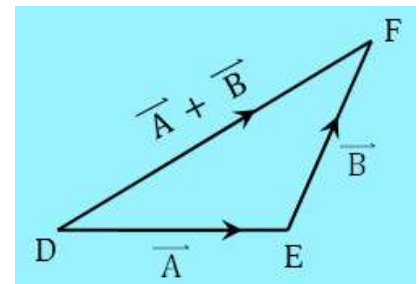
**Lesson (3-3)****Operations on Vectors****Lesson objectives****Related Links**

- ✚ Identify Adding vectors and their geometric representation.
- ✚ Recognize Triangle rule of adding vectors.
- ✚ Identify Parallelogram rule of adding vectors.
- ✚ Recognize Subtracting vectors and their graphic representation.
- ✚ Express the directed line segment in terms of the position vectors of its ends.

**THE ADDITION OF VECTORS GEOMETRICALLY****First method (the triangle rule):**

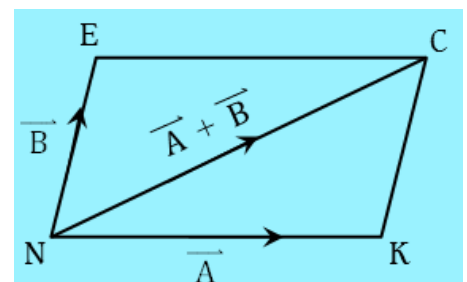
If $\overrightarrow{DE}$ represents $\vec{A}$, $\overrightarrow{EF}$ represents $\vec{B}$, then:
 $\overrightarrow{DF}$ represents the vector $\vec{A} + \vec{B}$

i.e. $\overrightarrow{DE} + \overrightarrow{EF} = \overrightarrow{DF}$

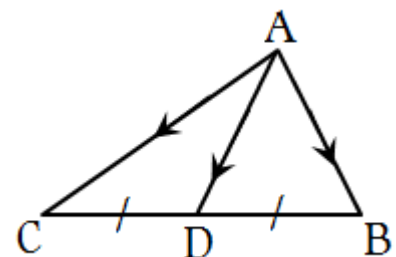
**Second method (the parallelogram rule):**

If $\overrightarrow{NK}$ represents $\vec{A}$, $\overrightarrow{NE}$ represents $\vec{B}$, then:
 $\overrightarrow{NC}$ represents the vector $\vec{A} + \vec{B}$

i.e. $\overrightarrow{NK} + \overrightarrow{NE} = \overrightarrow{NC}$

**Remarks**

- ① In any triangle we get: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$
- ② If $\overrightarrow{AD}$ is a median in ΔABC
 (as in the opposite figure), then $\overrightarrow{AB} + \overrightarrow{AC} = 2 \overrightarrow{AD}$



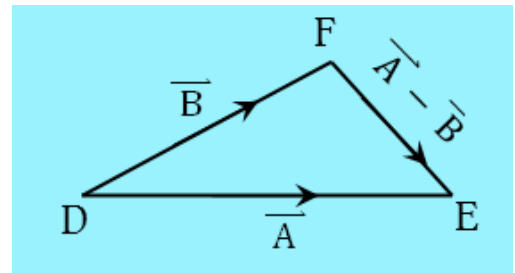


THE DIFFERENCE OF VECTORS GEOMETRICALLY

If $\overrightarrow{DE}$ represents $\vec{A}$, $\overrightarrow{DF}$ represents $\vec{B}$, then:

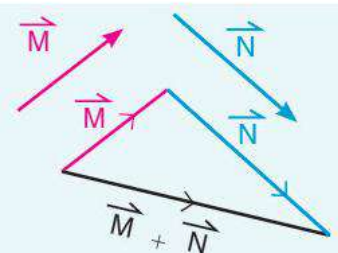
$\overrightarrow{FE}$ represents the vector $\vec{A} - \vec{B}$

i.e. $\overrightarrow{DE} - \overrightarrow{DF} = \overrightarrow{FE}$

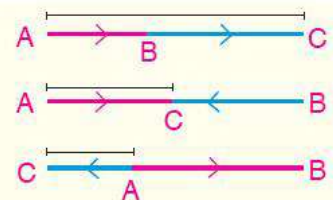


Important Notes:

- 1- Any two vectors $\vec{M}$ and $\vec{N}$ could be added (finding their resultant) by constructing two consecutive vectors equivalent to the two vectors $\vec{M}$ and $\vec{N}$ as in the figure opposite.



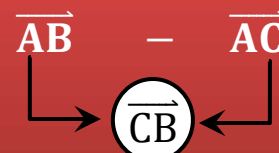
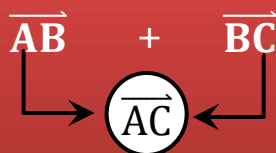
- 2- Shal rule of adding two vectors is true if the points A, B and C belong to a straight line.
In the three figures opposite, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



- 3- $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \vec{0}$ (Identity element of addition operation of vectors)
 $\therefore \overrightarrow{BA}$ is the additive inverse of the vector $\overrightarrow{AB}$
i.e. $\overrightarrow{BA} = -\overrightarrow{AB}$



Remember that:



**Example ①****Deduce that:**

- ① In any triangle we get: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$
- ② If $\overrightarrow{AD}$ is a median in ΔABC , then $\overrightarrow{AB} + \overrightarrow{AC} = 2 \overrightarrow{AD}$

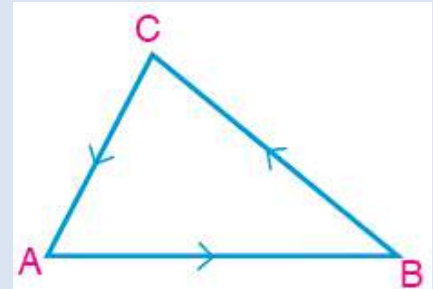
Solution① In ΔABC :

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$$

$$\therefore -\overrightarrow{AC} = \overrightarrow{CA}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

② In ΔABC :

$$\therefore \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD} \quad \rightarrow (1)$$

In ΔACD :

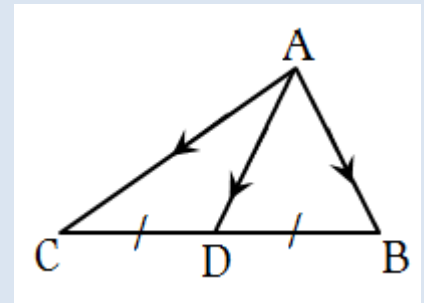
$$\therefore \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD} \quad \rightarrow (2)$$

By adding (1) & (2):

$$\therefore \overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{AC} + \overrightarrow{CD} = 2 \overrightarrow{AD}$$

$$\therefore \overrightarrow{BD} + \overrightarrow{CD} = \vec{0}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2 \overrightarrow{AD}$$



**Example 2**

In any quadrilateral, prove that:

$$\textcircled{1} \quad \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$$

$$\textcircled{2} \quad \overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{BC}$$

Solution

① Construction: Draw $\overrightarrow{BD}$

In $\triangle ABD$:

$$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD} \quad \rightarrow (1)$$

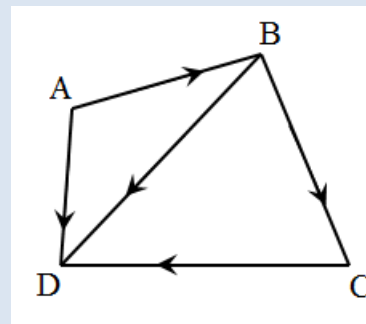
In $\triangle BCD$:

$$\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD} \quad \rightarrow (2)$$

By adding (1) & (2):

$$\therefore \overrightarrow{AB} + \cancel{\overrightarrow{BD}} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD} + \cancel{\overrightarrow{BD}}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$$



②

In $\triangle ACD$:

$$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD} \quad \rightarrow (1)$$

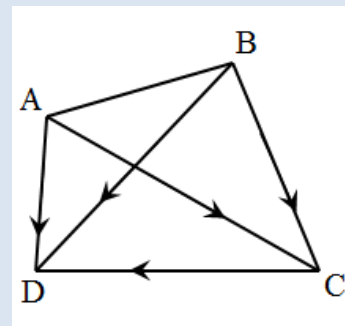
In $\triangle BCD$:

$$\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD} \quad \rightarrow (2)$$

By adding (1) & (2):

$$\therefore \overrightarrow{AC} + \cancel{\overrightarrow{CD}} + \overrightarrow{BD} = \overrightarrow{BC} + \cancel{\overrightarrow{CD}} + \overrightarrow{AD}$$

$$\therefore \overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{BC}$$



Remember that:

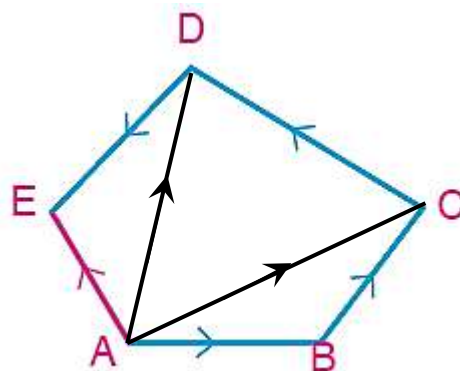
$$\overrightarrow{AB} = -\overrightarrow{BA}$$

**Example 3**

In the figure ABCDE:

Deduce that: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$

Proof



Construction: Draw $\overrightarrow{AC}$ & $\overrightarrow{AD}$

In ΔABC :

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad \rightarrow (1)$$

In ΔACD :

$$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD} \quad \rightarrow (2)$$

In ΔADE :

$$\overrightarrow{AD} + \overrightarrow{DE} = \overrightarrow{AE} \quad \rightarrow (3)$$

By adding (1), (2) & (3):

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} + \cancel{\overrightarrow{AC}} + \overrightarrow{CD} + \cancel{\overrightarrow{AD}} + \overrightarrow{DE} = \cancel{\overrightarrow{AC}} + \cancel{\overrightarrow{AD}} + \overrightarrow{AE}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$$

**Example 4**

In any quadrilateral ABCD, prove that: $\overrightarrow{AB} + \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{DB}$

Proof

In $\triangle ABC$:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad \rightarrow (1)$$

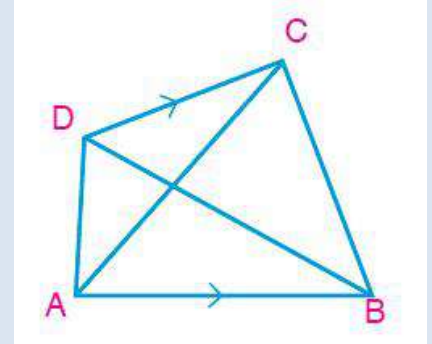
In $\triangle BCD$:

$$\overrightarrow{DB} + \overrightarrow{BC} = \overrightarrow{DC} \quad \rightarrow (2)$$

By adding (1) & (2):

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{DB} + \overrightarrow{BC}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{DB}$$

**Example 5**

ABCD is a quadrilateral in which $\overrightarrow{BC} = 3 \overrightarrow{AD}$.

Prove that: $\overrightarrow{AC} + \overrightarrow{BD} = 4 \overrightarrow{AD}$.

Proof

In $\triangle ACD$:

$$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD} \quad \rightarrow (1)$$

In $\triangle BCD$:

$$\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD} \quad \rightarrow (2)$$

By adding (1) & (2):

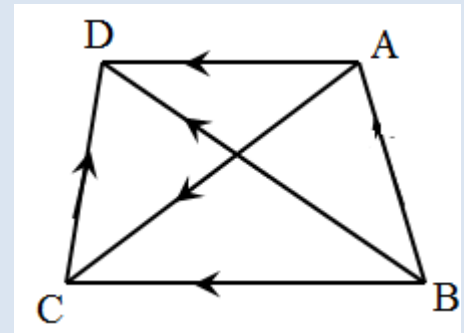
$$\therefore \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{AD}$$

$$\therefore \overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{BC}$$

$$\because \overrightarrow{BC} = 3 \overrightarrow{AD}$$

$$\therefore \overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AD} + 3 \overrightarrow{AD}$$

$$\therefore \overrightarrow{AC} + \overrightarrow{BD} = 4 \overrightarrow{AD}$$



**Example 6**

ABCD is a parallelogram, its diagonals are intersecting at M. N is a point in the same plane. **Prove that:**

(A) $\overrightarrow{AB} + \overrightarrow{AD} + 2\overrightarrow{CM} = \vec{0}$

(B) $\overrightarrow{NA} + \overrightarrow{NC} = \overrightarrow{NB} + \overrightarrow{ND}$

Proof

(A) $\because$ ABCD is a parallelogram & its diagonals intersecting at M

$\therefore$ M is mid point of $\overline{BD}$

In $\triangle ABD$:

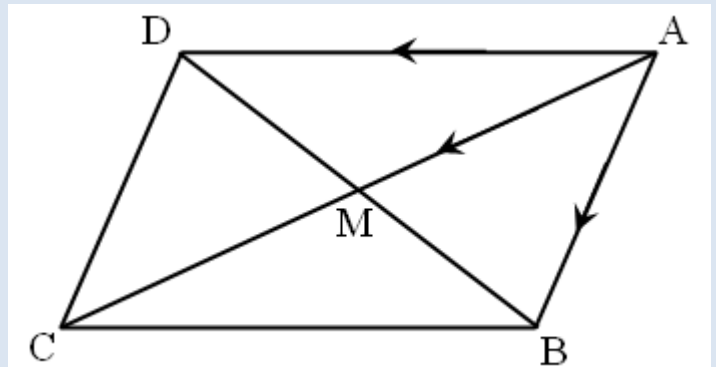
$$\therefore \overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AM}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{AD} - 2\overrightarrow{AM} = \vec{0}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{AD} + 2\overrightarrow{MA} = \vec{0}$$

$$\therefore \overrightarrow{MA} = \overrightarrow{CM}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{AD} + 2\overrightarrow{CM} = \vec{0}$$



(B)

In $\triangle ANB$:

$$\therefore \overrightarrow{NA} + \overrightarrow{AB} = \overrightarrow{NB} \quad \rightarrow (1)$$

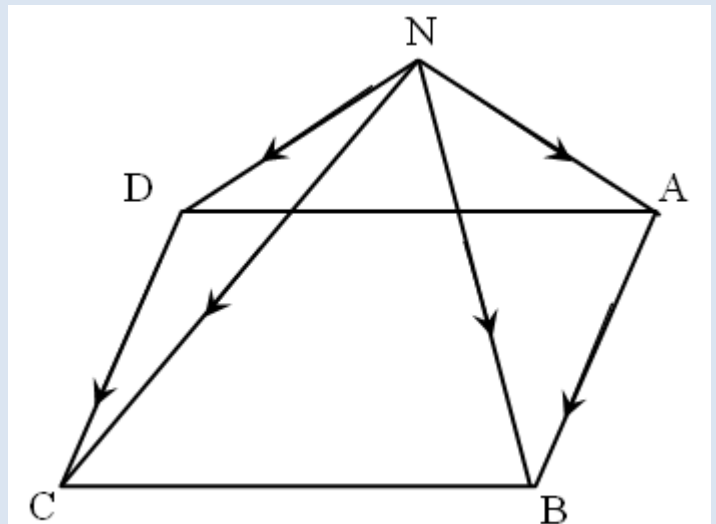
In $\triangle NCD$:

$$\therefore \overrightarrow{ND} + \overrightarrow{DC} = \overrightarrow{NC} \quad \rightarrow (2)$$

By adding (1) & (2):

$$\therefore \overrightarrow{NA} + \overrightarrow{AB} + \overrightarrow{NC} = \overrightarrow{NB} + \overrightarrow{ND} + \overrightarrow{DC}$$

$$\therefore \overrightarrow{NA} + \overrightarrow{NC} = \overrightarrow{NB} + \overrightarrow{ND}$$



**Example 7**

ABCD is a parallelogram in which E is the midpoint of $\overline{CB}$.

Prove that: $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{DC} = 2 \overrightarrow{AE}$

Proof

Construction: Draw $\overline{AC}$

In $\triangle ABC$:

$\because$ E is the midpoint of $\overline{BC}$

$$\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2 \overrightarrow{AE} \quad \rightarrow (1)$$

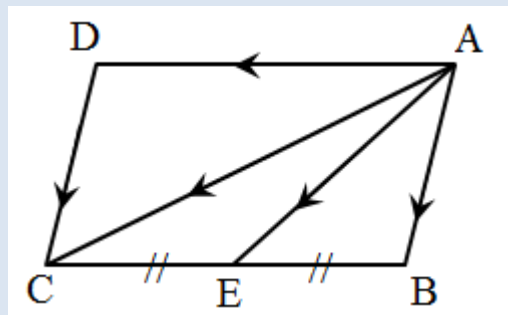
In $\triangle ADC$:

$$\therefore \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC} \quad \rightarrow (2)$$

By adding (1) & (2):

$$\overrightarrow{AB} + \cancel{\overrightarrow{AC}} + \overrightarrow{AD} + \overrightarrow{DC} = 2 \overrightarrow{AE} + \cancel{\overrightarrow{AC}}$$

$$\therefore \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{DC} = 2 \overrightarrow{AE}$$

**Remark**

If: $\overrightarrow{A} = (x_1, y_1)$, $\overrightarrow{B} = (x_2, y_2)$, then: $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (x_2 - x_1, y_2 - y_1)$

Definition of the norm of a vector (the length of the vector)

If: $\overrightarrow{A} = (x, y)$, then the real number $\sqrt{x^2 + y^2}$ is called the norm (or the length) of the vector $\overrightarrow{A}$ **i.e.** $\| \overrightarrow{A} \| = \sqrt{x^2 + y^2}$



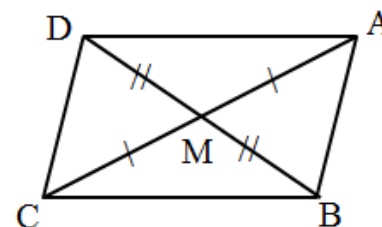
PRACTICE

Choose the correct answer

Q (1): In the opposite figure:

All the following statement expresses $\overrightarrow{AC}$ except the statement

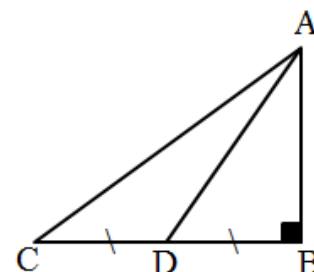
- (a) $2 \overrightarrow{AM}$
- (b) $\overrightarrow{AD} + \overrightarrow{DC}$
- (c) $\overrightarrow{AB} + \overrightarrow{BD}$
- (d) $\overrightarrow{AB} + \overrightarrow{AD}$



Q (2): In the opposite figure:

$2 \overrightarrow{AD} = \dots\dots\dots$

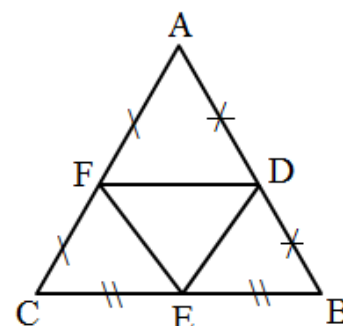
- (a) $2 \overrightarrow{AB} + 2 \overrightarrow{CD}$
- (b) $\overrightarrow{AB} + \overrightarrow{BD}$
- (c) $\overrightarrow{AB} + \overrightarrow{AC}$
- (d) $\overrightarrow{BA} + \overrightarrow{CA}$



Q (3): In the opposite figure:

All the following statement express $\overrightarrow{DF}$ except the statement

- (a) $\overrightarrow{DA} + \overrightarrow{AF}$
- (b) $\overrightarrow{ED} - \overrightarrow{EF}$
- (c) $\overrightarrow{EF} - \overrightarrow{ED}$
- (d) $\overrightarrow{DB} + \overrightarrow{BC} + \overrightarrow{CF}$



Q (4): ABCD is a parallelogram: $\overrightarrow{AC} \cap \overrightarrow{BD} = \{M\}$, then: $\overrightarrow{AB} + \overrightarrow{AD} = \dots\dots\dots$

- (a) $\overrightarrow{CA}$
- (b) $\overrightarrow{BD}$
- (c) $2 \overrightarrow{MC}$
- (d) $2 \overrightarrow{DM}$

Q (5): If ABCD is a rectangle, then $\overrightarrow{AC} + \overrightarrow{BD} = \dots\dots\dots$

- (a) $\overrightarrow{CD}$
- (b) $2 \overrightarrow{BA}$
- (c) $\overrightarrow{BC}$
- (d) $2 \overrightarrow{BC}$

Q (6): The additive inverse of $\overrightarrow{AB}$ is

- (a) $\overrightarrow{BA}$
 (c) $\vec{0}$
- (b) $\overrightarrow{AB}$
 (d) $-\overrightarrow{BA}$

Q (7): Which of the following equivalent to the zero vector?

- Ⓐ $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC}$ Ⓑ $\overrightarrow{CD} - \overrightarrow{DF} - \overrightarrow{FC}$
 Ⓒ $\overrightarrow{FE} - \overrightarrow{FD} - \overrightarrow{DE}$ Ⓓ $\overrightarrow{AB} - \overrightarrow{AN} + \overrightarrow{CN}$

Q (8): ABCDE is a regular pentagon, then: $\overrightarrow{DE} + \overrightarrow{EA} - \overrightarrow{BA} = \dots\dots\dots$

- Ⓐ $\overrightarrow{DB}$
 Ⓑ $\overrightarrow{AD}$
 Ⓒ $\overrightarrow{CE}$
 Ⓓ $\overrightarrow{BD}$

Q (9): If D is the midpoint of $\overline{BC}$, A is a point $\notin \overline{BC}$, then

- Ⓐ $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$
 Ⓑ $\overrightarrow{AB} + \overrightarrow{AC} = 2 \overrightarrow{AD}$
 Ⓒ $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} = \vec{0}$
 Ⓓ $\overrightarrow{AB} + \overrightarrow{AC} + 2 \overrightarrow{AD} = \vec{0}$

Q (10): In ΔABC , if D , E are the midpoints of $\overline{AB}$, $\overline{AC}$ and $\overline{AB} = \overline{M}$, $\overline{AC} = \overline{N}$, then $\overline{DE} = \dots\dots\dots$

- (a) $\vec{M} + \vec{N}$
 (b) $\vec{M} - \vec{N}$
 (c) $\frac{1}{2}(\vec{M} + \vec{N})$
 (d) $-\frac{1}{2}(\vec{M} - \vec{N})$

Q (11): ABCDEF is a regular hexagon, $\overrightarrow{AB} = \vec{M}$, $\overrightarrow{BC} = \vec{N}$, $\overrightarrow{CD} = \vec{F}$, then $\overrightarrow{AE} =$ (in terms of $\vec{M}, \vec{N}, \vec{F}$)

- Ⓐ $\vec{M} + \vec{N} + \vec{F}$
 Ⓑ $\vec{M} - \vec{N} + \vec{F}$
 Ⓒ $\vec{N} + \vec{F}$
 Ⓓ $\vec{F} - \vec{M}$

**Lesson (4-3)****Applications on Vectors****Lesson objectives****Related Links**

- ✚ Use the vectors and operation on them to prove some geometric theorems.
- ✚ Solve geometric applications in plane geometry using vectors.
- ✚ Solve physics applications on vectors.
- ✚ Find the resultant of many forces.
- ✚ Recognize Equilibrium of forces.
- ✚ Find Relative velocity.

**Example ①**

If $3\vec{N} - 2\vec{AB} = 3\vec{CB} + 5\vec{BA}$. **Prove that:** $\vec{N} = \vec{CA}$.

Solution

$$\begin{aligned}
 \because 3\vec{N} - 2\vec{AB} &= 3\vec{CB} + 5\vec{BA} & \therefore 3\vec{N} &= 3\vec{CB} + 5\vec{BA} + 2\vec{AB} \\
 \therefore 3\vec{N} &= 3\vec{CB} + 3\vec{BA} & \therefore 3\vec{N} &= 3(\vec{CB} + \vec{BA}) \\
 \therefore \vec{N} &= (\vec{CB} + \vec{BA}) & \therefore \vec{N} &= \vec{CA}
 \end{aligned}$$

Example ②

If $2\vec{M} + 3\vec{AB} = 2\vec{CB} - \vec{BA}$. **Prove that:** $\vec{M} = \vec{CA}$.

Solution

$$\begin{aligned}
 \because 2\vec{M} + 3\vec{AB} &= 2\vec{CB} - \vec{BA} & \therefore 2\vec{M} &= 2\vec{CB} - \vec{BA} - 3\vec{AB} \\
 \therefore 2\vec{M} &= 2\vec{CB} - \vec{BA} + 3\vec{BA} & \therefore 2\vec{M} &= 2\vec{CB} + 2\vec{BA} \\
 \therefore 2\vec{M} &= 2(\vec{CB} + \vec{BA}) & \therefore \vec{M} &= \vec{CA}
 \end{aligned}$$

**Example 3**

ABCD is a parallelogram where A (2, -1), B (7, 1), C (4, 4).

Find the coordinates of the point D.

Solution

$$\because \overline{AD} \parallel \overline{CB}, AD = BC$$

$$\therefore \overrightarrow{AD} = \overrightarrow{BC}$$

$$\therefore D - A = C - B$$

$$\therefore D = A + C - B$$

$$\therefore D = (2, -1) + (4, 4) - (7, 1) = (-1, 2)$$

Example 4

ABCD is a quadrilateral in which A (1, -2), B (9, 0), C (8, 4), D (0, 2).

Prove that: (A) $\overrightarrow{AB} = \overrightarrow{DC}$.

(B) $\overrightarrow{AB} \perp \overrightarrow{BC}$

Solution

$$(A) \because \overrightarrow{AB} = B - A = (9, 0) - (1, -2) = (8, 2)$$

$$\therefore \overrightarrow{DC} = C - D = (8, 4) - (0, 2) = (8, 2) \qquad \therefore \overrightarrow{AB} = \overrightarrow{DC}$$

$$(B) \because \overrightarrow{AB} = B - A = (9, 0) - (1, -2) = (8, 2)$$

$$\therefore \overrightarrow{BC} = C - B = (8, 4) - (9, 0) = (-1, 4)$$

$$\therefore m_1 (\text{slope of } \overrightarrow{AB}) = \frac{y}{x} = \frac{2}{8} = \frac{1}{4}$$

$$m_2 (\text{slope of } \overrightarrow{BC}) = \frac{y}{x} = \frac{4}{-1} = -4$$

$$\therefore m_1 \times m_2 = -1$$

$$\therefore \overrightarrow{AB} \perp \overrightarrow{BC}$$

**Example 5**

Use vectors to prove that: the points A (1, 4) , B (-1, -2) , C (2, -3) are vertices of right angled-triangle at B.

Solution

In ΔABC :

$$\therefore \overrightarrow{AB} = B - A = (-1, -2) - (1, 4) = (-2, -6)$$

$$\therefore m_1(\overrightarrow{AB}) = \frac{y}{x} = \frac{-6}{-2} = 3$$

$$\therefore \overrightarrow{CB} = B - C = (-1, -2) - (2, -3) = (-3, 1)$$

$$\therefore m_2(\overrightarrow{CB}) = \frac{y}{x} = \frac{1}{-3}$$

$$\therefore m_1 \times m_2 = -1$$

$$\therefore \overrightarrow{AB} \perp \overrightarrow{CB}$$

$\therefore ABC$ is right angled-triangle at B

PHYSICAL APPLICATIONS

1 Resultant Force The forces acting on an object are subjected to the process of adding vectors, the result of this operation is known as the resultant forces acting on the object where: $F = F_1 + F_2 + \dots$

Example 6

If the forces: $\overrightarrow{F_1} = 2\vec{i} + \vec{j}$, $\overrightarrow{F_2} = \vec{i} + 7\vec{j}$, $\overrightarrow{F_3} = \vec{i} - 5\vec{j}$ act on a particle, Calculate the magnitude and direction of their resultant (forces are measured in Newton).

Solution

$$\therefore F = F_1 + F_2 + F_3$$

$$\therefore \overrightarrow{F} = (2 + 1 + 1)\vec{i} + (1 + 7 - 5)\vec{j}$$

$$\therefore \overrightarrow{F} = 4\vec{i} + 3\vec{j}$$

$$\therefore \text{The magnitude of the resultant} = \|\overrightarrow{F}\| = \sqrt{4^2 + 3^2} = 5 \text{ Newton}$$

$$\therefore \text{Direction of the resultant } \theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{3}{4} \simeq 37^\circ$$

**Example 7**

If the forces: $\vec{F}_1 = (-6, 6)$, $\vec{F}_2 = (9, 13)$ and $\vec{F}_3 = (2, -7)$ act on a particle where the forces are measures in Dyne. **Find:** the magnitude and direction of the resultant of these forces.

Solution

$$\therefore \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$\therefore \vec{F} = (-6, 6) + (9, 13) + (2, -7) = (5, 12)$$

$$\therefore \vec{F} = 5\vec{i} + 12\vec{j}$$

$$\therefore \|\vec{F}\| = \sqrt{5^2 + 12^2} = 13 \text{ Dyne}$$

$$\therefore \tan \theta = \frac{y}{x} = \frac{12}{5}$$

$$\therefore \theta = 67^\circ 22' 48''$$

Example 8

If: $\vec{F}_1 = (a, b)$, $\vec{F}_2 = -3\vec{i} + 4\vec{j}$ act on a particle and system is in equilibrium, then **find** a and b

Solution

In equilibrium: the sum of x & y = 0

$$\therefore a - 3 = 0, \quad b + 4 = 0$$

$$\therefore a = 3, \quad b = -4$$

Example 9

Prove that the forces: $\vec{F}_1 = (-12, -8)$, $\vec{F}_2 = (10\sqrt{2}, \frac{\pi}{4})$ and $\vec{F}_3 = (2, 4)$ act on a particle are in balance.

Solution

$$\therefore \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$\therefore \vec{F} = (-12, -8) + (10\sqrt{2} \cos 45^\circ, 10\sqrt{2} \sin 45^\circ) + (2, 4)$$

$$\therefore \vec{F} = (-12 + 10 + 2, -8 + 10 + 4) = (0, 0)$$

$\therefore$ The forces are in equilibrium



② Relative Velocity

If: $\vec{V}_A$ is the actual velocity of car (A) and $\vec{V}_B$ is the actual velocity of car (B).

then:

$$\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$$

Example 10

A car (A) moves on a straight road with speed 70 km/h, A car (B) moves on the same road with speed 90 km/h. Find the relative velocity of car (A) with respect to car (B) when:

- (A) The two cars move in the same direction.
(B) The two cars move in the opposite direction.

Solution

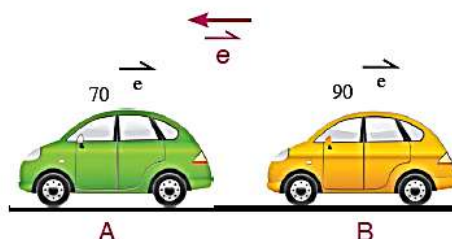
Consider $\vec{e}$ is the unit vector in the direction of movement of car (A)

- (A) The two cars move in the same direction:

$$\vec{V}_A = 70 \vec{e}$$

$$\vec{V}_B = 90 \vec{e}$$

$$\vec{V}_{AB} = \vec{V}_A - \vec{V}_B = 70 \vec{e} - 90 \vec{e} = -20 \vec{e}$$



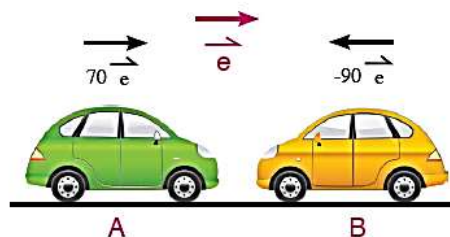
i.e. The observer in car (B) feels that the car (A) moves towards him with speed 20 km/h.

- (B) The two cars move in opposite direction:

$$\vec{V}_A = 70 \vec{e}$$

$$\vec{V}_B = -90 \vec{e}$$

$$\vec{V}_{AB} = \vec{V}_A - \vec{V}_B = 70 \vec{e} - (-90 \vec{e}) = 160 \vec{e}$$



i.e. the observer in car (B) feels that the car (A) moves towards him with speed 160 km/h.



PRACTICE

Choose the correct answer

Q (1): If $\vec{A} = 20\vec{i} - 15\vec{j}$, $\vec{B} = 7\vec{i} + 24\vec{j}$ and $\vec{M} = \vec{A} + \vec{B}$, $\vec{N} = \vec{A} - \vec{B}$, then

- Ⓐ $\vec{M} \parallel \vec{N}$
- Ⓑ $\vec{M} \perp \vec{N}$
- Ⓒ $\vec{M} = \vec{N}$
- Ⓓ $\|\vec{M}\| = \|\vec{N}\|$

Q (2): If $2\vec{AB} + 2\vec{BC} = k\vec{CA}$, then $k = \dots\dots\dots$

- Ⓐ 1
- Ⓑ -1
- Ⓒ 2
- Ⓓ -2

In the opposite graph:

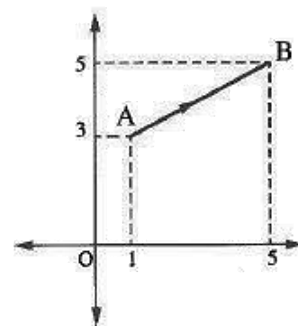
$$A = (1, 3)$$

$$, B = (5, 5)$$

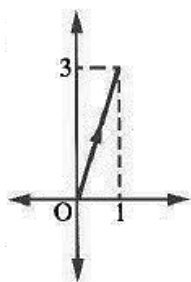
Q (3):

, then which of the

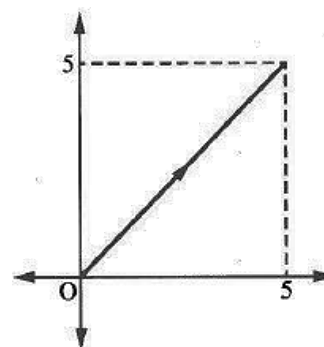
following represent $\vec{AB}$



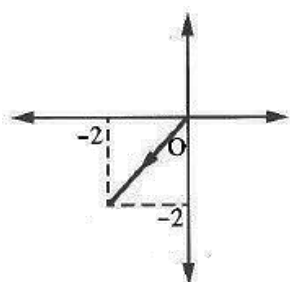
Ⓐ



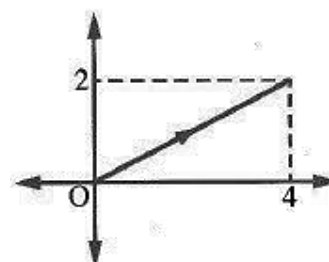
Ⓑ



Ⓒ



Ⓓ





Q (4): If:
 $\vec{F}_1 = (8\sqrt{2}, \frac{3}{4}\pi)$, $\vec{F}_2 = a\hat{i} + 3\hat{j}$, $\vec{F}_3 = -5\hat{i} + (b+2)\hat{j}$
 are acting in one point equilibrium, then $\frac{a}{b} = \dots\dots\dots$

- Ⓐ 13
- Ⓑ -13
- Ⓒ 1
- Ⓓ -1

Q (5): $\vec{F}_1 = \hat{i} - 3\hat{j}$, $\vec{F}_2 = 3\hat{i} + 6\hat{j}$
 , then the Force $\vec{F}_3$ which makes the result of the three forces is a unit vector and acts in the north direction =

- Ⓐ $-3\hat{i} - 3\hat{j}$
- Ⓑ $-4\hat{i} - 2\hat{j}$
- Ⓒ $-5\hat{i} - 3\hat{j}$
- Ⓓ $-4\hat{i} - 3\hat{j}$

Q (6): If:
 $\vec{V}_A = 120\hat{e}$, $\vec{V}_B = -80\hat{e}$, then $\vec{V}_{AB} = \dots\dots\dots\hat{e}$

- Ⓐ 200
- Ⓑ 40
- Ⓒ -40
- Ⓓ -200

Q (7): The magnitude of resultant of the forces acting on a body as trying to move it with a force of magnitude 70 newton and the magnitude of the force of friction is 55 newton equals newton

- | | |
|-------|------|
| Ⓐ 70 | Ⓑ 55 |
| Ⓒ 125 | Ⓓ 15 |



Q (8): If $\vec{F}_1 = i - 3j$, $\vec{F}_2 = 3i - j$ act on a particle, then the norm of the resultant equals newton

- (a) $2\sqrt{10}$
- (b) 8
- (c) $4\sqrt{2}$
- (d) 4

Q (9): If $\vec{V}_A = 12\vec{e}$, $\vec{V}_B = 8\vec{e}$, then $\vec{V}_{BA} = \dots\dots\dots$

- (a) $20\vec{e}$
- (b) $-20\vec{e}$
- (c) $4\vec{e}$
- (d) $-4\vec{e}$

Q (10): If $\vec{V}_A = 120\vec{e}$, $\vec{V}_B = -80\vec{e}$, then $\vec{V}_{AB} = \dots\dots\dots$

- (a) $40\vec{e}$
- (b) $200\vec{e}$
- (c) $-40\vec{e}$
- (d) $-200\vec{e}$

Q (11): If $\vec{V}_{AB} = 75\vec{e}$, $\vec{V}_A = 60\vec{e}$, then $\vec{V}_B = \dots\dots\dots$

- (a) $135\vec{e}$
- (b) $-135\vec{e}$
- (c) $15\vec{e}$
- (d) $-15\vec{e}$

Q (12): A car moved 20 meters in direction of North, then moved the same distance in the direction of west, then the displacement of the car is

- (a) 40 m in the west direction
- (b) 40 m in the western North direction
- (c) $20\sqrt{2}$ m in the western North direction
- (d) $20\sqrt{2}$ m in the western South direction



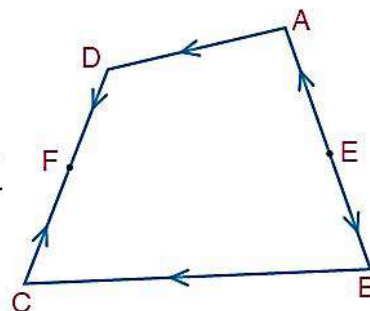
Home Work

① ABCD is a parallelogram, in which $A(3, 0)$, $B(0, 4)$, $D(-2, -1)$. Find the coordinates of the point C.

② In any triangle XYZ, Prove that: $\vec{XY} + \vec{YZ} + \vec{ZX} = \vec{0}$

③ In any quadrilateral ABCD, prove that: $\vec{AB} + \vec{BC} + \vec{CD} = \vec{AD}$.

④ In the figure opposite: ABCD is a quadrilateral, $E \in \overline{AB}$, $F \in \overline{CD}$.
prove that: $\vec{EB} + \vec{BC} + \vec{CF} = \vec{EA} + \vec{AD} + \vec{DF}$



⑤ ABCD is a quadrilateral, if $\vec{AC} + \vec{DB} = 2\vec{AB}$ prove that: ABCD is a parallelogram.

⑥ ABC is a triangle in which D is the mid-point of $\overline{AB}$, E is the mid-point of $\overline{AC}$.
Prove that: $\vec{AE} + \vec{CD} = \vec{EB} + \vec{DA}$.

⑦ In the triangle ABC: D, E and F are the mid-points of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ respectively. Prove that: $\vec{AE} + \vec{BF} = \vec{DC}$

⑧ ABCD is a trapezium in which $\overline{AD} \parallel \overline{BC}$, $\frac{AD}{BC} = \frac{2}{3}$. Prove that: $\vec{AC} + \vec{BD} = \frac{5}{2} \vec{AD}$

⑨ If $\vec{A} = 3\vec{i} - 2\vec{j}$, $\vec{B} = -\vec{i} - 4\vec{j}$

Find:

A $\vec{A} + \vec{B}$

B $\vec{A} - \vec{B}$

C $\|\vec{A} + \vec{B}\|$

D $2\vec{A} + 3\vec{B}$

E $\vec{A} - 3\vec{B}$

F $-3\vec{A}$

⑩ ABCD is a trapezium in which $A(-2, -3)$, $B(4, -1)$, $C(2, 5)$, $D(-1, k)$.

A If $\vec{AB} \parallel \vec{DC}$, find the value of k.

B Prove that: $\vec{CB} \perp \vec{AB}$.

C Find the area of the trapezium ABCD.

⑪ If $A(5, 1)$, $B(2, 5)$, $C(-2, 3)$, $D(-5, -4)$

Prove using vectors that the figure ABCD is a trapezium.



Unit 4

**Lesson (4-1)****Division of a line segment****Lesson objectives****Related Links**

- Identify Concept of division internally.
- Identify Concept of division externally.
- Find the ratio of division.

The internal division:

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

The external division:

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

Example ①

If $A = (3, -1)$, $B = (5, 2)$ and $C \in \overline{AB}$ such that: $2 AC = 3 CB$, find the coordinates of C which divides $\overline{AB}$ if:

① C divides $\overline{AB}$ internally.

② C divides $\overline{AB}$ externally.

Solution

$$\therefore 2 AC = 3 CB$$

$$\therefore \frac{AC}{CB} = \frac{3}{2}$$

$$\therefore m_1 = 3 \quad \& \quad m_2 = 2$$

$$\begin{aligned} \text{①} \quad x &= \frac{m_1x_2 + m_2x_1}{m_1 + m_2} = \frac{3 \times 5 + 2 \times 3}{3 + 2} = \frac{21}{5} \\ y &= \frac{m_1y_2 + m_2y_1}{m_1 + m_2} = \frac{3 \times 2 + 2 \times (-1)}{3 + 2} = \frac{4}{5} \end{aligned}$$

$$\therefore C = \left(\frac{21}{5}, \frac{4}{5} \right)$$

$$\begin{aligned} \text{②} \quad x &= \frac{m_1x_2 - m_2x_1}{m_1 - m_2} = \frac{3 \times 5 - 2 \times 3}{3 - 2} = 9 \\ y &= \frac{m_1y_2 - m_2y_1}{m_1 - m_2} = \frac{3 \times 2 - 2 \times (-1)}{3 - 2} = 8 \end{aligned}$$

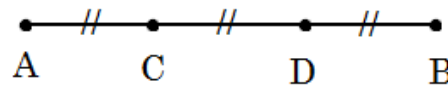
$$\therefore C = (9, 8)$$

**Example 2**

If: $A = (-1, 4)$ and $B = (5, -2)$, find the coordinates of the two points C and D that divide $\overline{AB}$ to three equal parts in length.

Solution

Let C (x, y) divide $\overline{AB}$ in ratio 1 : 2 from A



$$\therefore x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} = \frac{1 \times 5 + 2 \times -1}{1 + 2} = 1$$

$$\therefore y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} = \frac{1 \times -2 + 2 \times 4}{1 + 2} = 2$$

$$\therefore C = (1, 2)$$

Let D (x, y) be a midpoint of $\overline{CB}$

$$\therefore D = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right) = \left(\frac{5 + 1}{2}, \frac{-2 + 2}{2} \right) = (3, 0)$$

$$D = (3, 0) \therefore$$

Important remark

■ If we want to find the ratio and the type of division which a given point divides a line segment between two known points, we substitute in the law of internal to find $\frac{m_1}{m_2}$, then the division will be:

❶ **Internal** if $\frac{m_1}{m_2}$ is positive.

❷ **External** if $\frac{m_1}{m_2}$ is negative.

**Example 3**

Find the ratio in the line segment $\overline{AB}$ is divided by each of its points of intersection with the two coordinates axes if: $A = (4, -3)$ and $B = (-3, 5)$, then find the points of intersection.

Solution

Let $C(x, 0)$ the intersection point of $\overleftrightarrow{AB}$ with x -axis.

$$\therefore y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \qquad \therefore 0 = \frac{5m_1 - 3m_2}{m_1 + m_2}$$

$$\therefore 5m_1 - 3m_2 = 0 \qquad \therefore 5m_1 = 3m_2$$

$$\therefore \frac{m_1}{m_2} = \frac{3}{5} \quad (\text{Internal division})$$

$$\therefore x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} = \frac{5 \times -3 + 3 \times 4}{3 + 5} = -\frac{3}{8}$$

$$\therefore C = \left(-\frac{3}{8}, 0\right)$$

Let $D(0, y)$ the intersection point of $\overleftrightarrow{AB}$ with y -axis.

$$\therefore x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} \qquad \therefore 0 = \frac{-3m_1 + 4m_2}{m_1 + m_2}$$

$$\therefore -3m_1 + 4m_2 = 0 \qquad \therefore 3m_1 = 4m_2$$

$$\therefore \frac{m_1}{m_2} = \frac{4}{3} \quad (\text{Internal division})$$

$$\therefore y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} = \frac{4 \times 5 + 3 \times -3}{4 + 3} = \frac{11}{7}$$

$$D = \left(0, \frac{11}{7}\right) \therefore$$

**PRACTICE****Question 1**

If the coordinates of A and B are $(5, 5)$ and $(-1, -4)$ respectively, find the coordinates of the point C that divides $\overline{AB}$ internally by the ratio $2 : 1$.

- ☐ A $(3, 2)$
- ☐ B $(-1, 1)$
- ☐ C $(1, -1)$
- ☐ D $(-1, -1)$

Question 2

The coordinates of points A and B are $(4, 4)$ and $(1, -2)$ respectively. Given that $\overleftrightarrow{AB}$ intersects the x -axis at C and the y -axis at D , find the ratio by which $\overline{AB}$ is divided by points C and D respectively showing the type of division in each case.

- ☐ A internal division by ratio $1 : 4$, external division by ratio $1 : 2$
- ☐ B internal division by ratio $4 : 1$, external division by ratio $2 : 1$
- ☐ C internal division by ratio $1 : 2$, external division by ratio $1 : 4$
- ☐ D internal division by ratio $2 : 1$, external division by ratio $4 : 1$

Question 3

The coordinates of A and B are $(1, 9)$ and $(9, 9)$ respectively. Determine the coordinates of the points that divide $\overline{AB}$ into four equal parts.

- ☐ A $(9, 5), (9, 7), (-4, 2)$
- ☐ B $(5, 9), (7, 9), (3, 9)$
- ☐ C $(5, 9), (7, 9), (7, 5)$
- ☐ D $(5, 9), (7, 9), (-2, 0)$



Question 4

A bus is traveling from city $A(10, -10)$ to city $B(-8, 8)$. Its first stop is at C , which is halfway between the cities. Its second stop is at D , which is two-thirds of the way from A to B . What are the coordinates of C and D ?

- ☐ A $(0, 0), (-3, 3)$
- ☐ B $(1, -1), (-2, 2)$
- ☐ C $(1, -1), (4, -4)$
- ☐ D $(2, -2), (-2, 2)$

Question 5

Given $A(-5, 9)$ and $B(7, -3)$, what are the points C and D that divide $\overline{AB}$ into three parts of equal length?

- ☐ A $(-1, 5), (3, 1)$
- ☐ B $(\frac{2}{3}, 2), (\frac{23}{6}, -\frac{1}{2})$
- ☐ C $(1, 3), (1, 3)$
- ☐ D $(\frac{4}{3}, 4), (\frac{2}{3}, 2)$

Question 6

Consider $A(-1, -2)$ and $B(-7, 7)$. Find the coordinates of C , given that C is on the ray $\overrightarrow{AB}$ but NOT on the segment $\overline{AB}$ and $AC = 2CB$.

- ☐ A $(-3, 1)$
- ☐ B $(5, -11)$
- ☐ C $(-13, 16)$
- ☐ D $(-5, 4)$



Question 7

Consider points $A(2, 3)$ and $B(-4, -3)$. Find the coordinates of C , given that C is on the ray $\overrightarrow{BA}$ but NOT on the segment $\overline{AB}$ and $AC = 2AB$.

- ☐ A (14, 15)
- ☐ B (0, 1)
- ☐ C (-2, 3)
- ☐ D (8, 9)

Question 8

Given $A(6, -6)$ and $B(-7, -1)$, find the coordinates of C on $\overleftrightarrow{AB}$ for which $2AC = 9CB$.

- ☐ A (51, 21), (-75, 3)
- ☐ B $(-\frac{51}{7}, -3)$, (-75, 3)
- ☐ C (-51, -21), (-75, 3)
- ☐ D $(-\frac{51}{11}, -\frac{21}{11})$, $(-\frac{75}{7}, \frac{3}{7})$

Question 9

The coordinates of the points A and B are $(-3, 4)$ and $(-4, -2)$ respectively. Determine the coordinates of the point C , given that it divides $\overleftrightarrow{AB}$ externally in the ratio $2 : 1$.

- ☐ A (-2, 10)
- ☐ B (-8, -5)
- ☐ C (-5, -8)
- ☐ D (5, -8)



HOME WORK

Q1:

If the coordinates of A and B are $(5, 5)$ and $(-1, -4)$ respectively, find the coordinates of the point C that divides $\overline{AB}$ internally by the ratio $2 : 1$.

Q2:

If the coordinates of A and B are $(3, 1)$ and $(-7, 1)$ respectively, find the coordinates of the point C that divides $\overline{AB}$ internally by the ratio $2 : 3$.

Q3:

The coordinates of points A and B are $(4, 4)$ and $(1, -2)$ respectively. Given that $\overline{AB}$ intersects the x -axis at C and the y -axis at D , find the ratio by which $\overline{AB}$ is divided by points C and D respectively showing the type of division in each case.

Q4:

The coordinates of points A and B are $(6, 6)$ and $(1, -4)$ respectively. Given that $\overline{AB}$ intersects the x -axis at C and the y -axis at D , find the ratio by which $\overline{AB}$ is divided by points C and D respectively showing the type of division in each case.

Q5:

The coordinates of A and B are $(1, 9)$ and $(9, 9)$ respectively. Determine the coordinates of the points that divide $\overline{AB}$ into four equal parts.

Q6:

A bus is travelling from city $A(10, -10)$ to city $B(-8, 8)$. Its first stop is at C , which is halfway between the cities. Its second stop is at D , which is two-thirds of the way from A to B . What are the coordinates of C and D ?

Q7:

Given $A(-5, 9)$ and $B(7, -3)$, what are the points C and D that divide $\overline{AB}$ into three parts of equal length?

**Q8:**

Consider $A(-1, -2)$ and $B(-7, 7)$. Find the coordinates of C , given that C is on the ray $\overrightarrow{AB}$ but NOT on the segment $\overline{AB}$ and $AC = 2CB$.

Q9:

Consider points $A(2, 3)$ and $B(-4, -3)$. Find the coordinates of C , given that C is on the ray $\overrightarrow{BA}$ but NOT on the segment $\overline{AB}$ and $AC = 2AB$.

Q10:

Given $A(6, -6)$ and $B(-7, -1)$, find the coordinates of C on $\overrightarrow{AB}$ for which $2AC = 9CB$.

Q11:

The coordinates of the points A and B are $(-3, 4)$ and $(-4, -2)$ respectively. Determine the coordinates of the point C , given that it divides $\overrightarrow{AB}$ externally in the ratio $2 : 1$.

Q12:

The coordinates of the points A and B are $(2, 2)$ and $(5, 1)$ respectively. Determine the coordinates of the point C , given that it divides $\overrightarrow{AB}$ externally in the ratio $4 : 3$.

Q13:

Suppose $A(1, 3)$ and another point B , and that $C(5, 1)$ divides $\overrightarrow{AB}$ internally in the ratio $2 : 3$. What are the coordinates of B ?

Q14:

Line segment $\overline{AD}$ is a median in $\triangle ABC$, where $A = (8, -7)$ and $D = (2, -1)$. Find the point of intersection of the medians of the triangle ABC .

Q15:

Find the ratio by which the y -axis divides the line segment $\overline{AB}$, joining points $A(-3, 6)$ and $B(-8, -4)$, showing the type of division, and determine the coordinates of the point of division.



Revision on what have been studied

The equation of the straight line: $y = mx + c$

The slope of the straight line $m = \frac{y_2 - y_1}{x_2 - x_1}$

The slope of any horizontal straight line (parallel to x-axis) = 0.

The slope of any vertical straight line (parallel to y-axis) is undefined.

Important Remarks

- If $L_1 \parallel L_2$, then $m_1 = m_2$.
- If $L_1 \perp L_2$, then $m_1 \times m_2 = -1$ ($m_1 = \frac{-1}{m_2}$)
- The coordinate of midpoint = $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right)$
- The distance = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

**Lesson (4-2)****Equation of the straight line****Lesson objectives****Related Links**

- ✚ Find the equation of the straight line in terms of a given point and a direction vector.
- ✚ Find the general form of the equation of the straight line.
- ✚ Find the equation of the straight line in terms of the intercepted parts from the two axes.

**The slope of the straight line (m):**

- ① Which makes an angle (θ) with positive direction of the x-axis: $m = \tan \theta$
- ② Which passes through the points (x_1, y_1) , (x_2, y_2) : $m = \frac{y_2 - y_1}{x_2 - x_1}$
- ③ Whose equation is in the form: $\vec{r} = (x_1, y_1) + k(a, b)$: $m = \frac{b}{a}$
- ④ Whose equation is in the form: $ax + by + c = 0$: $m = -\frac{a}{b} = \frac{-\text{coeff. of } x}{\text{coeff. of } y}$

I- Cartesian Formula:

The equation of the straight line knowing its slope (m) and the intercepted part from y-axis (c):

$$y = mx + c$$

The equation of the straight line knowing its intersection with the two coordinate axes (a, 0) & (0, b):

$$\frac{x}{a} + \frac{y}{b} = 1$$

**Example ①**

Find the equation of the straight line whose slope = $-\frac{1}{2}$ and passes through the point $(1, -2)$

Solution

$$\therefore m = -\frac{1}{2}$$

$$\therefore y = mx + c$$

$$\therefore y = -\frac{1}{2}x + c$$

By substituting with the point: $(1, -2)$

$$\therefore -2 = -\frac{1}{2} \times 1 + c$$

$$\therefore c = -2 + \frac{1}{2} = -\frac{3}{2}$$

$$\therefore y = -\frac{1}{2}x - \frac{3}{2}$$

By multiply by 2

$$\therefore 2y = -x - 3$$

$$\therefore \boxed{2y + x + 3 = 0}$$

Example ②

Find the equation of the straight line passing through the two points $(4, -2)$ and $(5, 2)$

Solution

$$\therefore m = \frac{-2 - 2}{4 - 5} = 4$$

$$\therefore y = mx + c$$

$$\therefore y = 4x + c$$

By substituting with the point: $(5, 2)$

$$\therefore 2 = 4 \times 5 + c$$

$$\therefore c = 2 - 20 = -18$$

$$\therefore y = 4x - 18$$

$$\therefore \boxed{y - 4x + 18 = 0}$$

**Example ③**

Find the equation of the straight line that makes with the positive direction of the x -axis a positive angle of measure 135° and intercepts a part of length 7 length units from the positive direction of y -axis.

Solution

$$\because m = \tan 135^\circ = -1$$

$$\because c = 7$$

$$\because y = m x + c$$

$$\therefore \boxed{y = -x + 7}$$

Example ④

Find the equation of the straight line passing through the point $(1, 3)$ and it is perpendicular to the straight line whose slope is $\frac{3}{2}$

Solution

$$\because m = -\frac{2}{3}$$

$$\because y = m x + c$$

$$\therefore y = -\frac{2}{3} x + c$$

By substituting with the point: $(1, 3)$

$$\therefore 3 = -\frac{2}{3} \times 1 + c$$

$$\therefore c = 3 + \frac{2}{3} = \frac{11}{3}$$

$$\therefore y = -\frac{2}{3} x + \frac{11}{3}$$

By multiply by 3

$$\therefore 3 y = -2 x + 11$$

$$\therefore \boxed{3 y + 2 x - 11 = 0}$$

Example ⑤

Find the equation of the straight line which cut x -axis at the point $(3, 0)$ and y -axis at the point $(0, -5)$

Solution

$$\because \frac{x}{a} + \frac{y}{b} = 1$$

$$\therefore \frac{x}{3} + \frac{y}{-5} = 1$$

$$\therefore \frac{1}{3} x - \frac{1}{5} y = 1$$

By multiply by 15

$$\therefore \boxed{5 x - 3 y = 15}$$



EQUATION OF THE STRAIGHT LINE GIVEN A POINT BELONGING TO IT AND A DIRECTION VECTOR TO IT

II- Vector form:

To determine the equation of the straight line passing through the point A and the vector

$\vec{u}$ is a direction vector to it,

Let point B belong to the straight line L and the vectors.

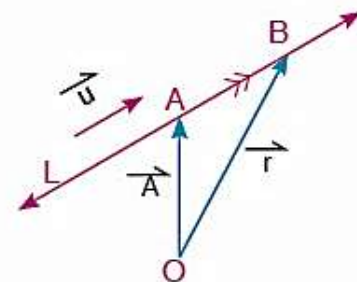
$\vec{r}$, $\vec{A}$ are represented to the two directed line segments

$\overrightarrow{OB}$, $\overrightarrow{OA}$ respectively, where O is any point in the plane.

then, there is a number $K \in \mathbb{R} - \{0\}$ where $\overrightarrow{AB} = \vec{r} - \vec{A} = K \vec{u}$

Then:

$$\vec{r} = \vec{A} + K \vec{u}$$



Example ⑥

Write the vector equation of the straight line which passes through the point $(-4, 3)$ and the vector $(2, 5)$ is its direction vector.

Solution

Let the straight line pass through point A $(-4, 3)$ and $\vec{u} = (2, 5)$

$\therefore \vec{r} = \vec{A} + K \vec{u}$ *vector form of the equation of the straight line.*

$\therefore$ The vector equation of the straight line is: $\boxed{\vec{r} = (-4, 3) + K (2, 5)}$



III- The parametric equations:

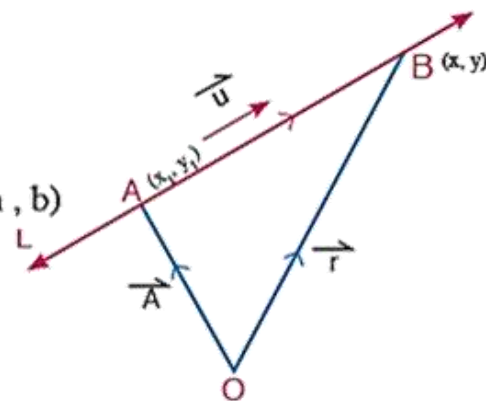
The vector equation is $\vec{r} = \vec{A} + K\vec{u}$

If $A(x_1, y_1)$, $B(x, y)$ are two points in the orthogonal coordinate system, O is the origin point and $\vec{u} = (a, b)$

then the equation of the straight line is $(x, y) = (x_1, y_1) + K(a, b)$

Then:

$$x = x_1 + k a, \quad y = y_1 + k b$$



Example 7

Write the parametric equations to the line passing through the point $(0, 5)$ and its direction vector is $(-1, 4)$.

Solution

Let $A(0, 5) \in L$, $\vec{u} = (-1, 4)$

$\therefore$ The vector equation of the line L is $\vec{r}(x, y) = (0, 5) + K(-1, 4)$ vector form

, then $x = 0 - K$, $y = 5 + 4K$

$\therefore x = -K$, $y = 5 + 4K$ are the parametric equations

Example 8

Find the Cartesian equation of the straight line which passes through the point $(2, 3)$ and its direction vector is $(4, -2)$

Solution

$$\therefore m = \frac{b}{a} = \frac{-2}{4} = -\frac{1}{2}$$

$$\therefore y = mx + c$$

$$\therefore y = -\frac{1}{2}x + c$$

By substituting with the point: $(2, 3)$

$$\therefore 3 = -\frac{1}{2} \times 2 + c$$

$$\therefore c = 3 + 1 = 4$$

$$\therefore y = -\frac{1}{2}x + 4$$

By multiply by 2

$$\therefore 2y + x = 8$$

**PRACTICE (1)****Question 1**

Write the equation of the line that passes through the points $(2, -2)$ and $(-2, 10)$ in the form $ax + by + c = 0$.

- ☐ A $4x + y - 4 = 0$
- ☐ B $4x + y + 4 = 0$
- ☐ C $3x + y + 4 = 0$
- ☐ D $3x - y - 4 = 0$
- ☐ E $3x + y - 4 = 0$

Question 2

Determine the area of the triangle bounded by the x -axis, the y -axis, and the straight line $2x + 7y + 28 = 0$.

- ☐ A 112 area units
- ☐ B 14 area units
- ☐ C 56 area units
- ☐ D 28 area units

Question 3

Write the equation of the line with slope $\frac{3}{2}$ and y -intercept $(0, 3)$ in the form $ax + by + c = 0$.

- ☐ A $3x - 2y + 3 = 0$
- ☐ B $3x - 2y - 6 = 0$
- ☐ C $3x - 2y + 6 = 0$
- ☐ D $x - 2y + 6 = 0$
- ☐ E $3x - y + 6 = 0$

**Question 4**

What are the x -intercept and y -intercept of the line $3x + 2y - 12 = 0$?

- ☐ A 2, 12
- ☐ B 3, 12
- ☐ C 4, 6
- ☐ D 2, 3
- ☐ E 3, 2

Question 5

A line passes through $(-5, -3)$ and cuts out a triangle of area 32 with the two coordinate axes. What is its equation?

- ☐ A $x + y + 8 = 0$
- ☐ B $x + y - 8 = 0$
- ☐ C $x - y + 8 = 0$
- ☐ D $x - y - 8 = 0$

Question 6

Find all the lines that pass through $(3, 2)$ and whose x - and y -intercepts combined distances from the origin is 12.

- ☐ A $2x + y + 8 = 0$
- ☐ B $2x + y - 8 = 0$
- ☐ C $x - 2y - 8 = 0$
- ☐ D $2x - y + 8 = 0$

**PRACTICE (2)****Question 1**

Which of the following can be the vector form of the equation of the straight line $ax + by + c = 0$, where $a \neq 0$ and $b \neq 0$?

- ☐ A $\vec{r} = \left(\frac{c}{a}, 0\right) + K(a, -b)$
- ☐ B $\vec{r} = \left(0, -\frac{c}{b}\right) + K(b, -a)$
- ☐ C $\vec{r} = \left(0, \frac{c}{b}\right) + K(b, -a)$
- ☐ D $\vec{r} = \left(-\frac{c}{a}, 0\right) + K(a, -b)$
- ☐ E $\vec{r} = \left(\frac{c}{a}, 0\right) + K(a, b)$

Question 2

Which of the following can be the vector form of the equation of the straight line $\frac{x}{a} + \frac{y}{b} = 1$, where $a \neq 0$ and $b \neq 0$?

- ☐ A $\vec{r} = (a, 0) + K(a, -b)$
- ☐ B $\vec{r} = \left(0, \frac{1}{a}\right) + K(-b, a)$
- ☐ C $\vec{r} = (0, b) + K(a, b)$
- ☐ D $\vec{r} = \left(0, \frac{1}{b}\right) + K(b, a)$
- ☐ E $\vec{r} = \left(-\frac{1}{a}, 0\right) + K(b, -a)$

Question 3

Find the vector equation of the straight line passing through the origin and the point $(0, 4)$.

- ☐ A $\vec{r} = (0, 4)$
- ☐ B $\vec{r} = \vec{K}(4, 0)$
- ☐ C $\vec{r} = (4, 0)$
- ☐ D $\vec{r} = \vec{K}(0, 4)$



Question 4

Find the vector equation of the straight line passing through the points $(6, -7)$ and $(-4, 6)$.

- ☐ A $\vec{r} = (-4, 6) + K(-13, 10)$
- ☐ B $\vec{r} = (-4, 6) + K(10, 13)$
- ☐ C $\vec{r} = (6, -7) + K(10, -13)$
- ☐ D $\vec{r} = (6, -4) + K(-7, 6)$

Question 5

Find the vector equation of the straight line whose slope is $-\frac{8}{3}$ and passes through the point $(4, -9)$.

- ☐ A $\vec{r} = (3, -8) + K(4, -9)$
- ☐ B $\vec{r} = (4, -9) + K(8, -3)$
- ☐ C $\vec{r} = (-9, 4) + K(3, -8)$
- ☐ D $\vec{r} = (4, -9) + K(3, -8)$

Question 6

Consider the line through the point $(3, 6)$ and perpendicular to vector $\vec{r} = (3, 6)$. Which of the following is a vector equation of this line?

- ☐ A $K = (3, 6) + \vec{r}(6, -3)$
- ☐ B $\vec{r} = (3, 6) + K(6, -3)$
- ☐ C $\vec{r} = (3, 6) + K(3, 6)$
- ☐ D $\vec{r} = (6, -3) + K(3, 6)$
- ☐ E $K = (3, 6) + \vec{r}(3, 6)$



Question 7

Write the vector equation of the straight line that passes through the point $(6, -9)$ with direction vector $(9, -2)$.

- ☐ A $\vec{r} = (9, -2) + k(6, -9)$
- ☐ B $\vec{r} = (6, -9) + k(9, -2)$
- ☐ C $\vec{r} = (-9, 2) + k(-6, 9)$
- ☐ D $\vec{r} = (-6, 9) + k(-9, 2)$

Question 8

Find the vector equation of the straight line that is parallel to the x -axis and passing through the point $(-5, 2)$.

- ☐ A $\vec{r} = (2, -5) + \vec{K}(1, 0)$
- ☐ B $\vec{r} = (-5, 2) + \vec{K}(1, 0)$
- ☐ C $\vec{r} = (2, -5) + \vec{K}(0, 1)$
- ☐ D $\vec{r} = (-5, 2) + \vec{K}(0, 1)$

Question 9

Consider the line through the point $(0, 4)$ and perpendicular to vector $\vec{r} = (0, 4)$. Which of the following is a vector equation of this line?

- ☐ A $K = (0, 4) + \vec{r}(4, 0)$
- ☐ B $\vec{r} = (0, 4) + K(4, 0)$
- ☐ C $\vec{r} = (0, 4) + K(0, 4)$
- ☐ D $\vec{r} = (4, 0) + K(0, 4)$
- ☐ E $K = (0, 4) + \vec{r}(0, 4)$

**PRACTICE (3)****Question 1**

Find the parametric equations of the straight line that passes through the point $(-9, 8)$ with direction vector $(4, -7)$.

- ☐ A $x = 8 - 7K, y = -9 + 4K$
- ☐ B $x = -9 + 8K, y = 4 - 7K$
- ☐ C $x = 8 + 4K, y = -9 - 7K$
- ☐ D $x = -9 + 4K, y = 8 - 7K$

Question 2

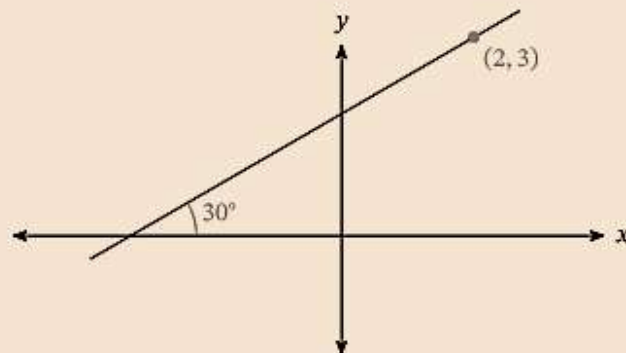
Find the parametric equations of the straight line that makes an angle of 135° with the positive x -axis and passes through the point $(1, -15)$.

- ☐ A $x = 1 + K, y = 1 - 15K$
- ☐ B $x = -15 - K, y = 1 + K$
- ☐ C $x = 1, y = -15 - K$
- ☐ D $x = 1 + K, y = -15 - K$

Question 3

Write a pair of parametric equations with parameter r describing the shown line.

- ☐ A $x = 2 + \frac{r}{2}, y = 3 - \frac{r\sqrt{3}}{2}$
- ☐ B $x = 2 - \frac{r}{2}, y = 3 + \frac{r\sqrt{3}}{2}$
- ☐ C $x = 2 - \frac{r}{2}, y = 3 - \frac{r\sqrt{3}}{2}$
- ☐ D $x = 2 + \frac{r\sqrt{3}}{2}, y = 3 + \frac{r}{2}$





HOME WORK

First: Complete each of the following

- ① If the straight line passes through the points $(3, 0)$, $(0, 2)$ is parallel to the straight line whose equation $y = ax - 3$, then a equals
- ② The vector equation of the straight line passes through the point $(3, 5)$ and parallel to the x -axis is
- ③ The cartesian equation of the straight line passes through the point $(-2, 7)$ and parallel to the y -axis is
- ④ The vector equation of the straight line passes through the origin point and the point $(1, 2)$ is
- ⑤ The equation of the straight line which makes 45° with the positive direction of the x -axis and cuts 5 units from the positive part of the y -axis is
- ⑥ The cartesian equation of the straight line which cuts the positive parts of the x -axis and the y -axis with magnitudes 2, 3 respectively is
- ⑦ Area of the triangle enclosed by the x -axis and the y - axis and the straight line $2x + 3y = 6$ equals

Second : Answer the following questions

- ① If the straight line whose equation $ax - 4y + 5 = 0$ makes an angle whose tangent is 0.75 with the positive direction of the x -axis , then find the value of a
- ② Find the vector equation of the straight line whose slope $\frac{1}{3}$ and passes through the point $(2, -1)$.
- ③ Find the parametric equations of the straight line which makes 45° with the positive direction of the x -axis and passes through the point $(3, -5)$.
- ④ Find the vector equation of the straight line which passes through the points $(2, -3)$, $(5, 1)$



⑤ Find the general equation of the straight line which passes through the points $(5, 0)$, $(0, -7)$

⑥ Find the Cartesian equation of the straight line which passing through the point $(3, -5)$ and parallel to the straight line whose equation is: $x + 2y - 7 = 0$

⑦ Find the vector equation of the straight line which passing through the point $(5, 7)$ and is perpendicular to the straight line whose equation is:

$$\vec{r} = (3, 0) + K (4, 3)$$

⑧ If $A (5, -6)$, $B (3, 7)$, $C (1, -3)$, find the equation of the straight line which passes through the point A and bisects $\overline{BC}$.

⑨ $\overline{AB}$ is a diameter in the circle M , if $B (-7, 11)$, $M (-2, 3)$, find the equation of the tangent to the circle at the point A .

**Lesson (4-3)****Measure of the angle between two straight lines****Lesson objectives****Related Links**

- Find the measure of the acute angle between two straight lines.

$$\tan \theta = \pm \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right) \text{ where } \theta \in [0^\circ, 180^\circ [$$

Example 1

Find the measure of the angle between the two straight lines:

$$L_1 : x - 2y + 5 = 0 \quad , \quad L_2 : 2x + 4y - 7 = 0$$

Solution

$$\therefore m_1 = \frac{1}{2}$$

$$\therefore m_2 = \frac{-2}{4} = -\frac{1}{2}$$

$$\therefore \tan \theta = \pm \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right) = \pm \left(\frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{1 + \frac{1}{2} \times \left(-\frac{1}{2}\right)} \right) = \pm \frac{4}{3}$$

$$\therefore \theta = \tan^{-1} \frac{4}{3} = 53^\circ 7' 48'' \quad (\text{acute})$$

$$\therefore \theta = 180^\circ - 53^\circ 7' 48'' = 126^\circ 52' 12'' \quad (\text{obtuse})$$

Important remarks**To determine the type of the triangle:**

Let AC is greatest side in the triangle ABC:

If $(AC)^2 > (AB)^2 + (BC)^2$ **then** $\angle B$ is an obtuse.

If $(AC)^2 = (AB)^2 + (BC)^2$ **then** $\angle B$ is a right.

If $(AC)^2 < (AB)^2 + (BC)^2$ **then** $\angle B$ is an acute.

**Example 2**

Find the measure of the angle between the two straight lines

$$\vec{r} = (2, 0) + k(-2, 1) \quad , \quad \vec{r} = (-3, 1) + k(6, 3)$$

Solution

$$\therefore m_1 = \frac{b}{a} = \frac{1}{-2} = -\frac{1}{2} \quad \& \quad \therefore m_2 = \frac{b}{a} = \frac{3}{6} = \frac{1}{2}$$

$$\therefore \tan \theta = \pm \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right) = \pm \left(\frac{\left(-\frac{1}{2}\right) - \frac{1}{2}}{1 + \left(-\frac{1}{2}\right) \times \frac{1}{2}} \right) = \pm \frac{4}{3}$$

$$\therefore \theta = \tan^{-1} \frac{4}{3} = 53^\circ 7' 48'' \quad (\text{acute})$$

$$\therefore \theta = 180^\circ - 53^\circ 7' 48'' = 126^\circ 52' 12'' \quad (\text{obtuse})$$

Example 3

Find the measure of the acute angle between the two straight lines whose equations are: $3x - 4y - 11 = 0$, $x + 7y + 5 = 0$

Solution

$$\therefore m_1 = \frac{3}{4}$$

$$\therefore m_2 = \frac{-1}{7}$$

$$\therefore \tan \theta = \pm \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right) = \pm \left(\frac{\frac{3}{4} - \left(\frac{-1}{7}\right)}{1 + \frac{3}{4} \times \left(\frac{-1}{7}\right)} \right) = \pm 1$$

$$\therefore \theta = \tan^{-1} 1 = 45^\circ \quad (\text{acute})$$

$$\therefore \theta = 180^\circ - 45^\circ = 135^\circ \quad (\text{obtuse})$$

**PRACTICE****Question 1**

Determine the measure of the acute angle between the straight line $x - y + 4 = 0$ and the straight line passing through the points $(3, -2)$ and $(-2, 4)$ to the nearest second.

- ☐ A $61^{\circ}23'22''$
- ☐ B $45^{\circ}0'0''$
- ☐ C $84^{\circ}48'20''$
- ☐ D $9^{\circ}27'44''$

Question 2

A line passing through $(8, 2)$ makes an angle θ with the line $6x + 4y + 9 = 0$, and $\tan \theta = \frac{15}{13}$. What is the equation of this line?

- ☐ A $71x - 69y + 410 = 0, -19x - 9y + 110 = 0$
- ☐ B $-71x - 9y + 214 = 0, -19x + 69y + 514 = 0$
- ☐ C $-69x + 71y + 410 = 0, -9x - 19y + 110 = 0$
- ☐ D $-9x - 71y + 214 = 0, -69x + 19y + 514 = 0$

Question 3

Determine, to the nearest second, the measure of the acute angle between two straight lines having slopes of 5 and $\frac{1}{4}$.

- ☐ A $75^{\circ}15'23''$
- ☐ B $66^{\circ}48'5''$
- ☐ C $64^{\circ}39'14''$
- ☐ D $76^{\circ}36'27''$

**Question 4**

Find, to the nearest second, the measure of the acute angle included between the straight line $6x - 7y + 40 = 0$ and the straight line whose slope is $\frac{7}{3}$.

- ☐ A $46^{\circ}45'45''$
- ☐ B $57^{\circ}55'4''$
- ☐ C $26^{\circ}12'0''$
- ☐ D $36^{\circ}25'51''$

Question 5

Find the measure of the acute angle between the two straight lines whose equations are $11x + 10y - 28 = 0$ and $2x + y + 15 = 0$ to the nearest second.

- ☐ A $15^{\circ}42'31''$
- ☐ B $44^{\circ}5'26''$
- ☐ C $54^{\circ}38'15''$
- ☐ D $22^{\circ}14'56''$

Question 6

Determine the measure of the acute angle between the two straight lines $L_1 : \vec{r} = (-4, -3) + K(4, -9)$ and $L_2 : 7x - 3y + 17 = 0$ to the nearest second.

- ☐ A $41^{\circ}7'17''$
- ☐ B $47^{\circ}9'40''$
- ☐ C $1^{\circ}7'24''$
- ☐ D $0^{\circ}54'34''$



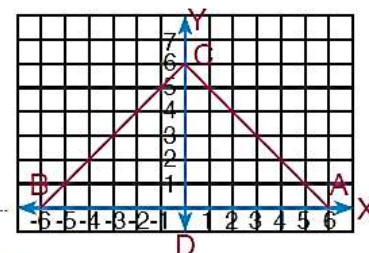
HOME WORK

First : Complete each of the following

- ① Measure of the angle between the straight lines whose slopes 2 , $-\frac{1}{2}$ equals
- ② Measure of the angle between the straight lines whose equation $x = 3$, $y = 4$ equals
- ③ Measure of the acute angle between the straight line whose equation $\vec{r} = (2, 2) + t(1, 1)$ and the straight line whose equation $x = 0$ equals
- ④ If the two straight lines whose equations $ax + 3y - 7 = 0$ and $2x - 3y + 5 = 0$ are parallel, then a equals
- ⑤ If the two straight lines whose equations $ax + 7y - 9 = 0$ and $7x - 2y + 12 = 0$ are perpendicular then a equals

Second :

The figure opposite shows a triangular piece of land, its vertices are $A(6, 0)$, $B(-6, 0)$, $C(0, 6)$. Complete each of the following:



- ⑥ Measure of the acute angle between $\vec{AC}$ and the x-axis equals
- ⑦ Measure of the angle between the straight lines $\vec{AC}$ and $\vec{BC}$ equals
- ⑧ The vector equation of the straight line $\vec{AC}$ is
- ⑨ The vector equation of the straight line $\vec{BC}$ is
- ⑩ The cartesian equation of the straight line which passes through the point C and parallel to $\vec{AB}$ is
- ⑪ Area of the triangle ABC equals

**Third : Choose the correct answer from the given answers**

- 12 Measure of the acute angle between the straight line which passes through the points $(0, 1)$, $(-1, 0)$ and the positive direction of the x-axis equals:
- (A) zero° (B) 45° (C) 60° (D) 90°
- 13 Measure of the acute angle between the straight line whose equation $\vec{r} = (0, 3) + t(1, 1)$ and the straight line whose equation $x = 0$ equals:
- (A) 30° (B) 45° (C) 60° (D) 90°
- 14 Measure of the acute angle between the straight lines whose equation $\sqrt{3}x - y = 4$ and $y = 3$ equals
- (A) 30° (B) 45° (C) 60° (D) 90°
- 15 The straight line perpendicular to the straight line whose equation $\vec{r} = (0, 5) + t(\sqrt{3}, 1)$ and makes an angle of measure with the positive direction of the x-axis
- (A) 30° (B) 60° (C) 120° (D) 150°

Fourth : Answer the following questions

- 16 Find the measure of the acute angle between each of the following pairs of the straight lines whose equations are:
- (A) $\vec{r} = (5, 0)$, $x - y + 4 = 0$
- (B) $\vec{r} = (0, 1) + t(1, 1)$, $2x - y - 3 = 0$
- (C) $y - \sqrt{3}x - 5 = 0$, $x - \sqrt{3}y - 6 = 0$
- 17 Prove that the triangle ABC is right at B where $A(5, 2)$, $B(2, -2)$, $C(-2, 1)$, then calculate the area of its surface.
- 18 If θ is the measure of the acute angle between the straight lines whose equations $x - 6y + 6 = 0$, $ax - 2y + 4 = 0$ and $\tan \theta = \frac{3}{4}$, then find the value of a.

**Lesson (4-4)****The length of the perpendicular from a point to a straight line****Lesson objectives****Related Links**

- Identify the length of a perpendicular from a given point to a straight line.

$ax + by + c = 0$, $P(x', y')$ is a given point in the Cartesian plane of this straight line:

$$\ell = \frac{|ax' + by' + c|}{\sqrt{a^2 + b^2}}$$

Example 1

Find the length of the perpendicular drawn from the point $N = (-2, 3)$ to the straight line $L : 3x - 4y + 9 = 0$

Solution

$$\ell = \frac{|ax' + by' + c|}{\sqrt{a^2 + b^2}} = \frac{|3(-2) - 4(3) + 9|}{\sqrt{3^2 + 4^2}} = \frac{9}{5}$$

Example 2

Find the length of the perpendicular drawn from the point $(5, 2)$ to the straight line which passes through the points $(0, -3)$, $(4, 0)$

Solution

$$\therefore \frac{x}{-3} + \frac{y}{4} = 1$$

By multiply by 12

$$\therefore -4x + 3y = 12$$

$$\therefore -4x + 3y - 12 = 0$$

$$\ell = \frac{|ax' + by' + c|}{\sqrt{a^2 + b^2}} = \frac{|-4 \times 5 + 3 \times 2 - 12|}{\sqrt{4^2 + 3^2}} = \frac{26}{5}$$

**Example ③**

Find the length of the perpendicular from the point (4, -5) to the straight line:

$$\vec{r} = (0, 2) + K (4, 3)$$

Solution

$$\therefore m = \frac{b}{a} = \frac{3}{4}$$

$$\therefore y = m x + c$$

$$\therefore y = \frac{3}{4}x + c$$

By substituting with the point: (0 , 2)

$$\therefore 2 = \frac{3}{4} \times 0 + c$$

$$\therefore c = 2$$

$$\therefore y = \frac{3}{4}x + 2$$

By multiply by 4

$$\therefore 4 y = 3 x + 8$$

$$\therefore 3 x - 4 y + 8 = 0$$

$$\ell = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}} = \frac{|3 \times 4 - 4 \times -5 + 8|}{\sqrt{3^2 + 4^2}} = \frac{40}{5} = 8$$

Example ④

Find the length of a perpendicular drawn from the point (2, -5) to the straight line: $\vec{r} = (-1, 0) + K (12, 5)$.

Solution

$$\therefore m = \frac{b}{a} = \frac{5}{12}$$

$$\therefore y = m x + c$$

$$\therefore y = \frac{5}{12}x + c$$

By substituting with the point: (-1 , 0)

$$\therefore 0 = \frac{5}{12} \times -1 + c$$

$$\therefore c = \frac{5}{12}$$

$$\therefore y = \frac{5}{12}x + \frac{5}{12}$$

By multiply by 12

$$\therefore 12 y = 5 x + 5$$

$$\therefore 5 x - 12 y + 5 = 0$$

$$\ell = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}} = \frac{|5 \times 2 - 12 \times -5 + 5|}{\sqrt{5^2 + 12^2}} = \frac{75}{13}$$

**PRACTICE****Question 1**

Determine the length of the perpendicular from a point $A(x_1, y_1)$ to the line $y = 0$.

- ☐ A $|y_1|$
- ☐ B $\frac{|y_1|}{|x_1|}$
- ☐ C $|x_1|$
- ☐ D $\sqrt{|x_1|^2 + |y_1|^2}$
- ☐ E 0

Question 2

Find the length of the perpendicular drawn from the origin to the straight line $-3x + 4y - 21 = 0$ rounded to the nearest hundredth.

- ☐ A 4.20 length units
- ☐ B 14.85 length units
- ☐ C 0.24 length units
- ☐ D 21.00 length units

Question 3

Find the length of the perpendicular from the point $(-22, -5)$ to the x -axis.

Question 4

Find the length of the perpendicular from the point $(-19, -13)$ to the y -axis.

**Question 5**

Find the length of the perpendicular drawn from the point $A(1, 9)$ to the straight line $-5x + 12y + 13 = 0$.

- ☐ A $\frac{116\sqrt{17}}{17}$ length units
- ☐ B $\frac{116}{13}$ length units
- ☐ C $\frac{116}{169}$ length units
- ☐ D $\frac{126}{13}$ length units

Question 6

Find the length of the perpendicular drawn from the point $A(-1, -7)$ to the straight line passing through the points $B(6, -4)$ and $C(9, -5)$.

- ☐ A $\frac{11\sqrt{10}}{5}$ units length
- ☐ B $\frac{\sqrt{10}}{16}$ units length
- ☐ C $\frac{8\sqrt{10}}{5}$ units length
- ☐ D $\frac{8\sqrt{2}}{5}$ units length

Question 7

If the length of the perpendicular drawn from the point $(-5, y)$ to the straight line $-15x + 8y - 5 = 0$ is 10 length units, find all the possible values of y .

- ☐ A $y = -\frac{43}{3}$ or $y = \frac{25}{3}$
- ☐ B $y = -30$ or $y = 30$
- ☐ C $y = -\frac{25}{2}$ or $y = \frac{25}{2}$
- ☐ D $y = -30$ or $y = \frac{25}{2}$

**Lesson (4–5)**

The general equation of the straight line passing through the point of intersection of two given lines

Example ①

Find the equation of the straight line which passes through the point of intersection of the two straight lines: $2x + 3y = 18$ and $5x - 2y - 7 = 0$ and passes through the point (5, 3)

Solution

$$\therefore 2x + 3y = 18 \quad \& \quad 5x - 2y = 7$$

$\therefore$ The intersection point of the two lines is: (3, 4)

The equation of the st. line that passing through the 2 points (3, 4) & (5, 3)

$$\therefore m = \frac{4 - 3}{3 - 5} = -\frac{1}{2}$$

$$\therefore y = mx + c \quad \therefore y = -\frac{1}{2}x + c$$

$$[\text{by substituting with any point}] \quad \therefore 4 = -\frac{1}{2} \times 3 + c \quad \therefore c = 4 + \frac{3}{2} = \frac{11}{2}$$

$$\therefore y = -\frac{1}{2}x + \frac{11}{2} \quad \text{Multiply by 2} \quad \therefore 2y = -x + 11$$

$$\therefore \text{The equation is: } \therefore \boxed{2y + x - 11 = 0}$$

Example ②

Find the equation of the straight line which passes through the point of intersection of the two straight lines: $3x + 2y = 10$ and $5x - 3y - 4 = 0$ and is perpendicular to the straight line: $2x + 7y - 4 = 0$

Solution

$$\therefore 3x + 2y = 10 \quad \& \quad 5x - 3y = 4$$

$\therefore$ The intersection point of the two lines is: (2, 2)

The straight line: $2x + 7y - 4 = 0$ its slope = $-\frac{2}{7}$

$$\therefore m = \frac{7}{2} \quad \therefore y = mx + c$$

$$\therefore y = \frac{7}{2}x + c \quad [\text{by substituting with the point (2, 2)}]$$

$$\therefore 2 = \frac{7}{2} \times 2 + c \quad \therefore c = 2 - 7 \quad \therefore c = -5$$

$$\therefore y = -\frac{1}{2}x + \frac{11}{2} \quad \text{Multiply by 2} \quad \therefore 2y = -x + 11$$

$$\therefore \text{The equation is: } \therefore \boxed{2y + x - 11 = 0}$$

**PRACTICE****Question 1**

Find the equation of the line perpendicular to $-6x - y + 8 = 0$ and passing through the intersection of the lines $-4x - y - 3 = 0$ and $-3x + 8y - 1 = 0$.

- ☐ A $173x - 158y + 101 = 0$
- ☐ B $2x - 3y + 1 = 0$
- ☐ C $7x - 42y - 1 = 0$
- ☐ D $19x - 39y + 8 = 0$

Question 2

Determine the equation of the line passing through the point of intersection of the two lines whose equations are $5x + 2y = 0$ and $3x + 7y + 13 = 0$ while making an angle of 135° with the positive y -axis.

- ☐ A $29x + y - 91 = 0$
- ☐ B $29x - 29y - 91 = 0$
- ☐ C $29x + 29y - 39 = 0$
- ☐ D $29x + 29y + 39 = 0$

Question 3

Find the equation of the straight line that passes through the origin and the point of intersection of the two straight lines $x = -\frac{17}{4}$ and $y = -5$.

- ☐ A $y = -\frac{20}{17}x$
- ☐ B $y = \frac{85}{4}x$
- ☐ C $y = \frac{20}{17}x$
- ☐ D $y = \frac{17}{20}x$

**Question 4**

Find the equation of the straight line that is parallel to the y -axis and passes through the point of intersection of the two straight lines $y = -3$ and $x = \frac{11}{15}y$.

- ☐ A $y = -\frac{11}{5}$
- ☐ B $x = -\frac{11}{5}$
- ☐ C $x = \frac{11}{5}$
- ☐ D $x = -\frac{11}{5}y$

Question 5

What is the equation of the line passing through $A(-1, 3)$ and the intersection of the lines $3x - y + 5 = 0$ and $5x + 2y + 3 = 0$?

- ☐ A $17x - 2y + 23 = 0$
- ☐ B $23x + 7y + 17 = 0$
- ☐ C $8x + y + 8 = 0$

Question 6

Find the equation of the straight line that passes through the point of intersection of the two straight lines $x - 8y = 2$ and $-6x - 8y = 1$ and is parallel to the y -axis.

- ☐ A $x = \frac{1}{7}$
- ☐ B $x = -\frac{1}{7}$
- ☐ C $x = -\frac{13}{56}$
- ☐ D $x = \frac{1}{7}y$



HOME WORK

- ① Find the vector equation of the straight line which passes through the origin point and the two straight lines whose equation $x = 3$, $y = 4$
.....
- ② Find the vector equation of the straight line which passes through the point $(3, 1)$, and the point of intersection of the two lines whose equations $3x + 2y - 7 = 0$, $x + 3y = 7$
- ③ Find the equation of the straight line passes through the point of intersection of the two straight lines whose equations $\vec{r} = k(-3, 2)$, $3x - 2y = 13$ and parallel to the y-axis.....
- ④ Find the equation of the straight line passes through the point of intersection of the two lines whose equations $2x + y = 5$, $x + 5y = 16$ and perpendicular to the line whose equation $x - y = 8$
- ⑤ Find the equation of the straight line passes through the point of intersection of the two lines whose equations $2x - 7y + 9 = 0$, $3x + 2y - 4 = 0$ and perpendicular to the first line. .
- ⑥ Find the equation of the line passes through the point of intersection of the two lines whose equations $2x + 3y - 2 = 0$, $3x - y - 14 = 0$ and makes an angle of measure 135° with the positive direction of the y-axis



General Exercises

First : Complete each of the following:

- ① The slope of the line whose equation $2x - 3y + 5 = 0$ equals
- ② The direction vector perpendicular to the line whose equation $\vec{r} = (2, -1) + t(3, -5)$ is ..
- ③ The equation of the line which makes an angle of measure 135° with the positive direction of the x-axis and passes through the point $(4, 0)$ is
- ④ The line whose equation $3x + 4y - 12 = 0$ intersects the two axes at the points
- ⑤ Measure of the acute angle between the line passes through $(0, 2)$, $(-2, 0)$ and the line whose equation $y = 0$ equals

Second; Multiple choice:

- ⑥ Measure of the acute angle between the two lines whose equations: $x - 3y + 5 = 0$, $x + 2y - 7 = 0$ equals
 (A) 15° (B) 30° (C) 45° (D) 60°
- ⑦ The equation of the line which passes through the point $(2, -3)$ and parallel to the x-axis is:
 (A) $x + 3 = 0$ (B) $y + 3 = 0$ (C) $x - 2 = 0$ (D) $y - 3 = 0$
- ⑧ Length of perpendicular drawn from the origin to the line whose equation $3x - 4y - 15 = 0$ equals
 (A) 3 (B) 4 (C) 5 (D) 15
- ⑨ All the following equations represent the equation of the line which passes through the points $(3, 0)$, $(0, 2)$ except the equation:
 (A) $\vec{r} = (3, 0) + t(3, -2)$ (B) $\vec{r} = (0, 2) + t(3, -2)$
 (C) $\vec{r} = (3, 0) + t(2, 3)$ (D) $\vec{r} = (0, 2) + t(-6, 4)$
- ⑩ Length of perpendicular drawn from the point $(0, -5)$ to the straight line whose equation $x + 7 = 0$ equals
 (A) 2 (B) 5 (C) 7 (D) 12



Third:

- ⑪ If $A(5, 2)$, $B(-3, 1)$, find the ratio by which $\overrightarrow{AB}$ is divided by the x -axis, showing the type of division.
- ⑫ Find the length of perpendicular drawn from $A(2, 5)$ to the line whose equation $\overrightarrow{r} = (1, -2) + t(3, 4)$
- ⑬ Prove that the triangle ABC is right at B where $A(5, 2)$, $B(2, -2)$, $C(-2, 1)$ then calculate its area

Unit test

First : complete questions

- ① The direction vector perpendicular to the line whose equation $8x - 7y + 12 = 0$ is
- ② If the two lines whose equations: $2x + by + 3 = 0$, $\overrightarrow{r} = (0, 2) + t(1, 3)$ are parallel , then b equals
- ③ The vector equation of the line passes through the origin and the point $(2, 1)$ is
- ④ The parametric equations of the line which makes an angle of measure 45° with the positive direction of the x -axis and passes through $(0, 5)$ is
- ⑤ Equation of the line which passes through the origin and the point of intersection of the two lines whose equations $x = 3$, $y = 3$ is

Second : multiple choice questions:

- ① Mid point of $\overline{AB}$ where $A(3, 7)$, $B(1, 5)$ is:
☐ A $(3, 5)$ ☐ B $(5, -2)$ ☐ C $(2, 5)$ ☐ D $(2, 6)$
- ② Equation of the line passes through the point $(2, -3)$ and parallel to the x -axis is:
☐ A $x = -3$ ☐ B $y = -3$ ☐ C $x = 2$ ☐ D $y = 3$
- ③ Measure of the acute angle between the two lines whose equation $x - 3y + 5 = 0$, $x + 2y - 7 = 0$ equals:
☐ A 45° ☐ B 60° ☐ C 120° ☐ D 135°
- ④ Length of perpendicular from the origin to the line whose equation $\overrightarrow{r} = (5, 0) + t(4, 3)$ equals
☐ A 3 ☐ B 4 ☐ C 5 ☐ D 15



- 5 Equation of the line passing through the points (3, 0), (0, 2) is:
- A $2x + 3y = 6$ B $3x + 2y = 6$ C $2x + 3y = 0$ D $3y + 2x = 0$

Third : Answer the following questions:

- 1 A ABCD is a parallelogram in which A(2, 2), B(3, 8), C(8, 10), find the coordinates of the point D.
- B Find the equation of the line passes through the point (7, 4) and parallel to the line whose equation $3x = 2y$.
- 2 A Prove that the points A (1, 4), B (3, -2), C (-3, 16) are collinear, then find the ratio by which the point B divides $\overline{AC}$, showing the type of division .
- B Prove that the two lines $L_1 : 2x + y - 4 = 0$, $L_2 : 2x + y + 2 = 0$ are parallel, and find the distance between them .
- 3 A Find the length of perpendicular from (8, -2) to the line whose equation: $\vec{r} = (0, \frac{1}{3}) + t(-3, 4)$.
- B If A (-1, 4), B (5, -2), find the coordinates of the point C which divides $\overline{AB}$ internally by the ratio 1 : 2.
- 4 A If A (2, 4), B (-7, 5), C (-3, 0), D (-2, 9) are the vertices of a square. Find the area of its surface.
- B Find the equation of the line passing through the point of intersection of the two lines $2x + y = 5$, $x - y = 1$ and the point (5, 3).



Revision on Unit (4)

- ① Find the measure of the acute angle between the two lines $x + 2y - 5 = 0$, $x - 3y + 2 = 0$.
- ② If $A(3, -2)$, $B(-2, 4)$, find the ratio by which $C(8, y)$ divides $\overrightarrow{AB}$, then find the value of y .
- ③ Prove that the triangle whose vertices $A(-4, 2)$, $B(1, -2)$, $C(1, 6)$ is an isosceles, then find its area.
- ④ Find the equation of the line passes through the point $(2, 5)$ and perpendicular to the line whose equation $2x + 3y + 7 = 0$.
- ⑤ Prove that the two lines whose equations $\overrightarrow{r} = (0, \frac{7}{3}) + t(3, 4)$ and $4x - 3y + 17 = 0$ are parallel , then find the distance between them.
- ⑥ Find the equation of the line passing through the point of intersection of the two lines $2x + 3y - 5 = 0$, $x + 3y = 7$ and the point $(2, 4)$.
- ⑦ Prove that the points $A(-2, 5)$, $B(3, 3)$, $C(-4, 2)$ are non collinear.
If $D(-9, 4)$, prove that the figure $ABCD$ is a parallelogram.
- ⑧ If $L_1: 3x + 4y - 5 = 0$, $L_2: \overrightarrow{r} = (0, 3) + t(1, -1)$, find each of :
 - Ⓐ Measure of the acute angle between L_1, L_2 .
 - Ⓑ Find the length of perpendicular drawn from $(0, -2)$ to the line L_1 .
- ⑨ Prove that the points $A(2, 3)$, $B(-1, -1)$, $C(3, -4)$, $D(6, 0)$ are the vertices of a square, then the area of its surface.
- ⑩ ABC is a triangle in which $A(1, 2)$, $B(5, 2)$, $C(7, 5)$, find
 - Ⓐ The cartesian equation to the line $\overrightarrow{BC}$.
 - Ⓑ The length of perpendicular from A to $\overrightarrow{BC}$.
 - Ⓒ The area of the triangle ABC .
- ⑪ Prove that the line: $3x - 4y + 3 = 0$ touches each of the two circles whose centres are $N_1(5, 2)$, $N_2(-2, 3)$ and whose radii are of the lengths 2, 3 unit of length respectively.
Are the two circles lie on one side or on two sides of the straight line?